Standard and Non–Standard Quantum Models : A Non–Commutative Version of the Classical System of SU(2) and SU(1,1) Arising from Quantum Optics

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Abstract

This is a challenging paper including some review and new results.

Since the non–commutative version of the classical system based on the compact group $SU(2)$ has been constructed in (quant-ph/0502174) by making use of Jaynes–Commings model and so–called Quantum Diagonalization Method in (quant-ph/0502147), we construct a non–commutative version of the classical system based on the non–compact group $SU(1,1)$ by modifying the compact case.

In this model the Hamiltonian is not hermite but pseudo hermite, which causes a big difference between two models. For example, in the classical representation theory of $SU(1,1)$, unitary representations are infinite dimensional from the starting point. Therefore, to develop a unitary theory of non–commutative system of $SU(1,1)$ we need an infinite number of non–commutative systems, which means a kind of **second non–commutativization**. This is a very hard and interesting problem.

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We develop a corresponding theory though it is not always enough, and present some challenging problems concerning how classical properties can be extended to the non–commutative case.

This paper is arranged for the convenience of readers as the first subsection is based on the standard model ($SU(2)$ system) and the next one is based on the non–standard model ($SU(1,1)$ system). This contrast may make the similarity and difference between the standard and non–standard models clear.

1 Introduction

This is a challenging paper including some review of [1] and new results, and our ultimate aim is to construct a unified theory of Non–Commutative (Differential) Geometry and Quantum Computation.

The Hopf bundles (which are famous examples of fiber bundles) over $\mathbf{K} = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ (the field of quaternion numbers), $\mathbf{O}$ (the field of octanion numbers) are classical objects and they are never written down in a local manner. If we write them locally then we are forced to encounter singular lines called the Dirac strings, see [1], [2].

It is very interesting to comment that the Hopf bundles correspond to topological solitons called Kink, Monopole, Instanton, Generalized Instanton respectively, see for example [2], [3], [4]. Therefore they are very important objects to study.

Berry has given another expression to the Hopf bundle and Dirac strings by making use of a Hamiltonian (a simple spin model including the parameters x, y and z), see the paper(s) in [5]. We call this the Berry model for simplicity. In this paper let us restrict to the case of $\mathbf{K}$=$\mathbf{C}$.

We also construct a pseudo Berry model by replacing the Pauli matrices (the generators of $su(2)$) in the Hamiltonian with the generators of $su(1,1)$. For this model the "Hamiltonian" is not hermite and a bundle is defined, which is called the pseudo Hopf bundle for simplicity. However, it is topologically trivial (therefore, there are no Dirac strings).

We would like to make the Hopf and pseudo Hopf bundles non–commutative. Whether such a generalization is meaningful or not is not clear at the current time, however it is worth trying,

see for example [6], [7] or more recently [8] and its references.

By the way, we are studying a quantum computation based on Cavity QED and one of the basic tools is the Jaynes–Cummings model (or more generally the Tavis–Cummings one), [9], [10], [11], [12]. This is given as a "half" of the Dicke model under the resonance condition and rotating wave approximation associated to it. If the resonance condition is not taken, then this model gives a non–commutative version of the Berry model. However, this new one is different from usual one because x and y coordinates are quantized, while z coordinate is not.

We also construct a non–commutative version of the pseudo Berry model by replacing the generators as in the classical case. In this case, since the eigenvalues of the pseudo Hamiltonian should be real, the domain is extremely limited in the Fock space.

From the non–commutative Berry model we construct a non–commutative version of the Hopf bundle by making use of so–called Quantum Diagonalization Method developed in [13]. Then we see that the Dirac strings appear in only states containing the ground one $(\mathcal{F} \times \{|0\rangle\} \cup \{|0\rangle\} \times \mathcal{F})$, while they don't appear in excited states $(\mathcal{F} \times \mathcal{F} - \mathcal{F} \times \{|0\rangle\} \cup \{|0\rangle\} \times \mathcal{F})$, where $\mathcal{F}$ is the Fock space generated by $\{a,\ a^\dagger,\ N = a^\dagger a\}$,

This means that classical singularities are not universal in the process of non–commutativization, which is a very interesting phenomenon. This is one of reasons why we consider non–commutative generalizations (which are not necessarily unique) of classical geometry.

We also construct a non–commutative version of the pseudo Hopf bundle in the non–commutative pseudo Berry model. Since in this case the bundle is trivial and there are no Dirac strings, the situation becomes easy.

Moreover, we construct a non–commutative version of the Veronese mapping which is the mapping from $\mathbf{C}P^1$ to $\mathbf{C}P^n$ with mapping degree n. The mapping degree is usually defined by making use of the (first–) Chern class, so our mapping will become important if a non–commutative (or quantum) "Chern class" would be constructed.

We also construct a non–commutative version of the pseudo Veronese mapping which is the mapping from $\mathbf{C}Q^1$ to $\mathbf{C}Q^n$ with mapping degree n.

We challenge to construct a non–commutative version of the spin representation of group

$SU(2)$. However, our trial is not enough because we could not construct the general case except for the special cases of spin $j = 1$ and $j = 3/2$. In this problem, we meet a difficulty coming from the non–commutativity. Further study constructing a general theory will be required.

We also challenge to construct a non–commutative version of the spin representation of group $SU(1, 1)$. However, unitary representations are infinite dimensional from the starting point even in the classical case. To develop a unitary theory of non–commutative system of $SU(1, 1)$ we need an infinite number of non–commutative systems, which means a kind of second non–commutativization. Therefore our trial is not enough, so that further study will be required.

Why do we consider non–commutative versions of classical field models ? What is an advantage to consider such a generalization ? Such natural questions arise. This paper may give one of answers. Moreover, readers will find many interesting (challenging) problems.

For the convenience of readers this paper is arranged as the first subsection is the system based on $SU(2)$ and the next one is the system based on $SU(1, 1)$. This contrast may make the similarity and difference between the standard and non–standard models clear. We also add many appendices to make the text clear.

The contents of the paper are as follows :

2 Mathematical Preliminaries

In this section we prepare some mathematical preliminaries for the following sections.

2.1 Classical $SU(2)$ System $\cdots$ Compact Case

The compact Lie group $SU(2)$ and its Lie algebra $isu(2)$ ($i = \sqrt{-1}$) are

$$SU(2) = \left\{ A \in M(2;\mathbf{C}) \mid A^{\dagger}A = 1_2, \quad \det(A) = 1 \right\} \tag{1}$$

and

$$su(2) = \left\{ X \in M(2;\mathbf{C}) \mid X^{\dagger} = X, \quad \mathrm{tr}(X) = 0 \right\}. \tag{2}$$

The algebra is generated by the famous Pauli matrices σ_j $(j = 1 \sim 3)$

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}; \quad 1_2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix},$$

and the map $su(2) \longrightarrow SU(2)$ is given as

$$\sum_{j=1}^{3} x_j \sigma_j \longrightarrow \exp\left(i \sum_{j=1}^{3} x_j \sigma_j\right).$$

We usually use

$$\sigma_+ \equiv (1/2)(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- \equiv (1/2)(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then the $su(2)$ relation

$$[\tilde{\sigma}_3, \sigma_+] = \sigma_+, \quad [\tilde{\sigma}_3, \sigma_-] = -\sigma_-, \quad [\sigma_+, \sigma_-] = 2\tilde{\sigma}_3$$

is well–known, where $\tilde{\sigma}_3 = (1/2)\sigma_3$.

Let us note that

$$A = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1 \tag{3}$$

is an element in $SU(2)$.

2.2 Classical $SU(1,1)$ System $\cdots$ Non–Compact Case

The non–compact Lie group $SU(1,1)$ and its Lie algebra $isu(1,1)$ are

$$SU(1,1) = \left\{B \in M(2; \mathbf{C}) \mid B^\dagger J B = J, \quad \det(B) = 1\right\} \tag{4}$$

and

$$su(1,1) = \left\{Y \in M(2; \mathbf{C}) \mid Y^\dagger = JYJ, \quad \mathrm{tr}(Y) = 0\right\} \tag{5}$$

where $J = \sigma_3$. The algebra is generated by the matrices τ_j $(j = 1 \sim 3)$

$$\tau_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} & -i \\ -i & \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \sigma_3,$$

and the map $su(1,1) \longrightarrow SU(1,1)$ is given as

$$\sum_{j=1}^{3} x_j \tau_j \longrightarrow \exp\left(i \sum_{j=1}^{3} x_j \tau_j\right).$$

We usually use

$$\tau_+ \equiv (1/2)(\tau_1 + i\tau_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \tau_- \equiv (1/2)(\tau_1 - i\tau_2) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Then the $su(1,1)$ relation

$$[\tilde{\tau}_3, \tau_+] = \tau_+, \quad [\tilde{\tau}_3, \tau_-] = -\tau_-, \quad [\tau_+, \tau_-] = -2\tilde{\tau}_3$$

is well–known, where $\tilde{\tau}_3 = (1/2)\tau_3$.

Let us note that

$$B = \begin{pmatrix} \alpha & -\bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1 \tag{6}$$

is an element in $SU(1,1)$.

3 Standard and Non–Standard Berry Models and Dirac Strings

We explain the way which Berry used in [5] to construct the Hopf bundle and Dirac strings corresponding to the compact case, and next construct ones corresponding to the non–compact case.

3.1 Standard Berry Model and Dirac Strings

The Hamiltonian used by Berry is a simple spin model

$$H_B = x\sigma_1 + y\sigma_2 + z\sigma_3 = (x - iy)\sigma_+ + (x + iy)\sigma_- + z\sigma_3 = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \tag{7}$$

where x, y and z are parameters. This Hamiltonian is of course hermite. We would like to diagonalize H_B above. The eigenvalues are

$$\lambda = \pm r \equiv \pm\sqrt{x^2+y^2+z^2}$$

and corresponding orthonormal eigenvectors are

$$|r\rangle = \frac{1}{\sqrt{2r(r+z)}}\begin{pmatrix} r+z \\ x+iy \end{pmatrix}, \quad |-r\rangle = \frac{1}{\sqrt{2r(r+z)}}\begin{pmatrix} -x+iy \\ r+z \end{pmatrix}.$$

Here we assume $(x,y,z) \in \mathbf{R}^3 - \{(0,0,0)\} \equiv \mathbf{R}^3 \setminus \{0\}$ to avoid a degenerate case. Therefore a unitary matrix defined by

$$A_I = (|r\rangle, |-r\rangle) = \frac{1}{\sqrt{2r(r+z)}}\begin{pmatrix} r+z & -x+iy \\ x+iy & r+z \end{pmatrix} \tag{8}$$

makes H_B diagonal like

$$H_B = A_I \begin{pmatrix} r & \\ & -r \end{pmatrix} A_I^\dagger \equiv A_I D_B A_I^\dagger. \tag{9}$$

We note that the unitary matrix A_I is not defined on the whole space $\mathbf{R}^3 \setminus \{0\}$. The defining region of U_I is

$$D_I = \mathbf{R}^3 \setminus \{0\} - \{(0,0,z) \in \mathbf{R}^3 |\ z < 0\}. \tag{10}$$

The removed line $\{(0,0,z) \in \mathbf{R}^3 |\ z < 0\}$ is just the (lower) Dirac string, which is impossible to add to D_I.

Next, we have another diagonal form of H_B like

$$H_B = A_{II} D_B A_{II}^\dagger \tag{11}$$

with the unitary matrix A_{II} defined by

$$A_{II} = \frac{1}{\sqrt{2r(r-z)}}\begin{pmatrix} x-iy & -r+z \\ r-z & x+iy \end{pmatrix}. \tag{12}$$

The defining region of A_{II} is

$$D_{II} = \mathbf{R}^3 \setminus \{0\} - \{(0,0,z) \in \mathbf{R}^3 |\ z > 0\}. \tag{13}$$

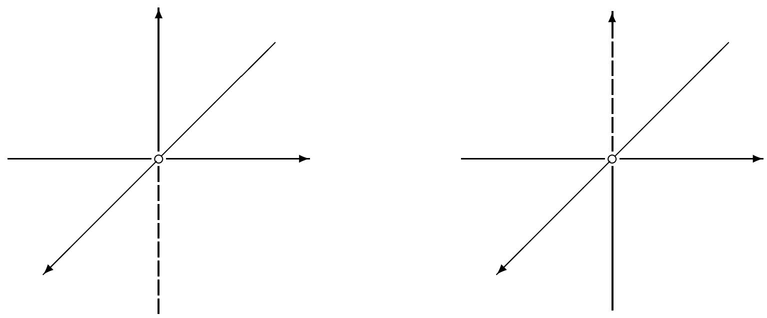

Figure 1: Dirac strings corresponding to I and II

The removed line $\{(0,0,z) \in \mathbf{R}^3 |\ z > 0\}$ is the (upper) Dirac string, which is also impossible to add to D_{II}.

Here we have diagonalizations of two types for H_B, so a natural question comes about. What is a relation between A_I and A_{II} ? If we define

$$\Phi = \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} x - iy & \\ & x + iy \end{pmatrix} \tag{14}$$

then it is easy to see

$$A_{II} = A_I \Phi.$$

We note that Φ (which is called a transition function) is not defined on the whole z–axis.

What we would like to emphasize here is that the diagonalization of H_B is not given globally (on $\mathbf{R}^3 \setminus \{0\}$). However, the dynamics is perfectly controlled by the system

$$\left\{(A_I, D_I), (A_{II}, D_{II}), \Phi, D_I \cup D_{II} = \mathbf{R}^3 \setminus \{0\}\right\}, \tag{15}$$

which defines a famous fiber bundle called the Hopf bundle associated to the complex numbers $\mathbf{C}$ [1],

$$S^1 \longrightarrow S^3 \longrightarrow S^2,$$

[1] The base space $\mathbf{R}^3 \setminus \{0\}$ is homotopic to the two–dimensional sphere S^2

see [2].

The projector corresponding to the Hopf bundle is given as

$$P(x,y,z) = A_I P_0 A_I^\dagger = A_{II} P_0 A_{II}^\dagger = \frac{1}{2r}\begin{pmatrix} r+z & x-iy \\ x+iy & r-z \end{pmatrix}, \tag{16}$$

where P_0 is the basic one

$$P_0 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \ \in \ M(2, \mathbf{C}).$$

It is well–known that P satisfies the relations

$$1)\ P^2 = P, \quad 2)\ P = P^\dagger, \quad 3)\ \mathrm{tr}P = 1.$$

We note that in (16) Dirac strings don't appear because the projector P is expressed globally.

3.2 Non–Standard Berry Model

The "Hamiltonian" that we consider here is a modified one of the Berry model

$$H_{pB} = x\tau_1 + y\tau_2 + z\tau_3 = (x-iy)\tau_+ + (x+iy)\tau_- + z\tau_3 = \begin{pmatrix} z & x-iy \\ -(x+iy) & -z \end{pmatrix} \tag{17}$$

where x, y and z are parameters. This is not hermite. As a tentative terminology we call this a pseudo Berry model. We would like to diagonalize H_{pB}. The eigenvalues are

$$\lambda = \pm s \equiv \pm\sqrt{z^2 - x^2 - y^2},$$

so the defining domain is

$$D \equiv \left\{ (x,y,z) \in \mathbf{R}^3 \mid z^2 - x^2 - y^2 > 0 \right\}$$

Here, to avoid a degenerate case of eigenvalues we removed the case of $z^2 - x^2 - y^2 = 0$. We note that D is not connected and consists of two domains D_+ and D_- defined by

$$D_+ = \{(x,y,z) \in D \mid z > 0\} \quad \text{and} \quad D_- = \{(x,y,z) \in D \mid z < 0\}. \tag{18}$$

The corresponding orthonormal eigenvectors are

$$|s\rangle = \frac{1}{\sqrt{2s(s+z)}}\begin{pmatrix} r+z \\ -(x+iy) \end{pmatrix}, \quad |-s\rangle = \frac{1}{\sqrt{2s(s+z)}}\begin{pmatrix} -x+iy \\ s+z \end{pmatrix}.$$

Therefore a matrix defined by

$$B_I = (|s\rangle, |-s\rangle) = \frac{1}{\sqrt{2s(s+z)}}\begin{pmatrix} s+z & -x+iy \\ -(x+iy) & s+z \end{pmatrix} \tag{19}$$

makes H_{pB} diagonal like

$$H_{pB} = B_I \begin{pmatrix} s & \\ & -s \end{pmatrix} B_I^{-1} \equiv B_I D_{pB} B_I^{-1}. \tag{20}$$

We note that the matrix B_I is an element of the non–compact group $SU(1,1)$, and is not defined on D_- because $s+z<0$. Moreover, B_I is defined on the whole $D_I = D_+$, so there is no singular line like Dirac strings.

A comment is in order. We have another diagonal form of H_{pB} like

$$H_{pB} = B_{II} D_{pB} B_{II}^{-1} \tag{21}$$

with the matrix B_{II} in $SU(1,1)$ defined by

$$B_{II} = \frac{1}{\sqrt{2s(z-s)}}\begin{pmatrix} -x+iy & z-s \\ z-s & -(x+iy) \end{pmatrix}. \tag{22}$$

The defining region of B_{II} is

$$D_{II} = D_+ - \{(0,0,z) \in D_+\}. \tag{23}$$

The removed line $\{(0,0,z) \in D_+\}$ is the (upper) Dirac string, which is also impossible to add to D_{II}.

However, with a singular transformation Φ (not defined on the whole z–axis) defined by

$$\Phi = \frac{1}{\sqrt{x^2+y^2}}\begin{pmatrix} -x-iy & \\ & -x+iy \end{pmatrix} \tag{24}$$

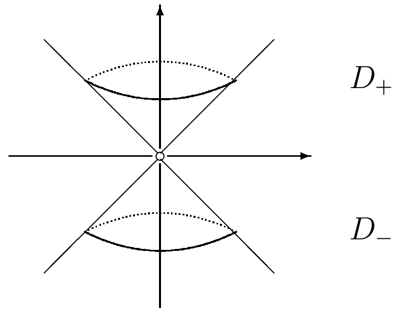

Figure 2: The domains D_+ and D_-

we can remove it because

$$B_I = B_{II}\Phi.$$

What we would like to emphasize in this case is that the diagonalization of H_{pB} is given globally on D_I, which is very different from the compact case.

$$\{H_{pB}, B_I, D_I\}\,. \tag{25}$$

Here, as a tentative terminology we call this system a **pseudo Hopf bundle** corresponding to the Hopf bundle in the compact case. However, this doesn't define a topological object because the domain D_I is contractible (trivial in the sense of topology).

The projector corresponding to the case is given as

$$Q(x,y,z) = B_I Q_0 B_I^{-1} = \frac{1}{2s}\begin{pmatrix} z+s & x-iy \\ -(x+iy) & -z+s \end{pmatrix}, \tag{26}$$

where $Q_0 = P_0$.

Q satisfies the relations

$$1)\ Q^2 = Q, \quad 2)\ JQJ = Q^\dagger, \quad 3)\ \mathrm{tr}Q = 1.$$

In the following we omit the suffix I for simplicity.

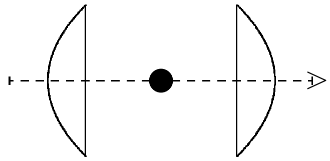

Figure 3: One atom and a single photon inserted in a cavity

4 Non–Commutative Models Arising from the Jaynes–Cummings Model

In this section let us explain the Jaynes–Cummings model which is well–known in quantum optics, see [9], [10]. From this we obtain a standard quantum model which is a natural extension of the (classical) Berry model. On the other hand, we obtain a non–standard quantum model by replacing the bases of $su(2)$ with the bases of $su(1,1)$.

4.1 Standard Quantum Model

The Hamiltonian of Jaynes–Cummings model can be written as follows (we set $\hbar = 1$ for simplicity)

$$H = \omega 1_2 \otimes a^\dagger a + \frac{\Delta}{2}\sigma_3 \otimes \mathbf{1} + g\left(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger\right), \tag{27}$$

where ω is the frequency of single radiation field, Δ the energy difference of two level atom, a and $a^\dagger$ are annihilation and creation operators of the field, and g a coupling constant. We assume that g is small enough (a weak coupling regime). See the figure 3 as an image of the Jaynes–Cummings model (we don't repeat here).

Now we consider the evolution operator of the model. We rewrite the Hamiltonian (27) as follows.

$$H = \omega 1_2 \otimes a^\dagger a + \frac{\omega}{2}\sigma_3 \otimes \mathbf{1} + \frac{\Delta - \omega}{2}\sigma_3 \otimes \mathbf{1} + g\left(\sigma_+ \otimes a + \sigma_- \otimes a^\dagger\right) \equiv H_1 + H_2. \tag{28}$$

Then it is easy to see $[H_1, H_2] = 0$, which leads to $\mathrm{e}^{-itH} = \mathrm{e}^{-itH_1}\mathrm{e}^{-itH_2}$.

In the following we consider e^{-itH_2} in which the resonance condition $\Delta - \omega = 0$ is not taken.

For simplicity we set $\theta=\frac{\Delta-\omega}{2g}(\neq 0)$ [2] then

$$H_2=g\left(\sigma_+\otimes a+\sigma_-\otimes a^\dagger+\frac{\Delta-\omega}{2g}\sigma_3\otimes\mathbf{1}\right)=g\left(\sigma_+\otimes a+\sigma_-\otimes a^\dagger+\theta\sigma_3\otimes\mathbf{1}\right).$$

For further simplicity we set

$$H_{JC}=\sigma_+\otimes a+\sigma_-\otimes a^\dagger+\theta\sigma_3\otimes\mathbf{1}=\begin{pmatrix}\theta & a\\ a^\dagger & -\theta\end{pmatrix},\quad [a,a^\dagger]=\mathbf{1} \tag{29}$$

where we have written θ in place of $\theta\mathbf{1}$ for simplicity.

H_{JC} can be considered as a non-commutative version of H_B under the correspondence $a\longleftrightarrow x-iy,\ a^\dagger\longleftrightarrow x+iy$ and $\theta\longleftrightarrow z$:

$$H_B=\begin{pmatrix}z & x-iy\\ x+iy & -z\end{pmatrix},\ [x-iy,x+iy]=0\longrightarrow H_{JC}=\begin{pmatrix}\theta & a\\ a^\dagger & -\theta\end{pmatrix},\ [a,a^\dagger]=\mathbf{1}. \tag{30}$$

That is, x and y coordinates are quantized, while z coordinate is not, which is different from usual one, see for example [7]. It may be possible for us to call this **a non–commutative Berry model**. We note that this model is derived not “by hand” but by the model in quantum optics itself.

4.2 Non–Standard Quantum Model

Similarly, from (29) we can define

$$H_{pJC}=\tau_+\otimes a+\tau_-\otimes a^\dagger+\theta\tau_3\otimes\mathbf{1}=\begin{pmatrix}\theta & a\\ -a^\dagger & -\theta\end{pmatrix},\quad [a,a^\dagger]=\mathbf{1} \tag{31}$$

by replacing $\{\sigma_j\}$ with $\{\tau_j\}$. In this case this model is derived “by hand”. It satisfies the $su(1,1)$ like relation (see (5))

$$\mathbf{J}H_{pJC}\mathbf{J}=H_{pJC}^\dagger\quad;\quad \mathbf{J}=\begin{pmatrix}\mathbf{1} & \\ & -\mathbf{1}\end{pmatrix}=J\otimes\mathbf{1}.$$

[2]Since the Jaynes–Cummings model is obtained by the Dicke model under some resonance condition on parameters included, it is nothing but an approximate one in the neighborhood of the point, so we must assume that $|\theta|$ is small enough. However, as a model in mathematical physics there is no problem to take θ be arbitrary

H_{pJC} can be considered as a non-commutative version of H_{pB} under the correspondence $a \longleftrightarrow x-iy,\ a^{\dagger} \longleftrightarrow x+iy$ and $\theta \longleftrightarrow z$:

$$H_{pB}=\left(\begin{array}{cc} z & x-iy \\ -(x+iy) & -z \end{array}\right),\ [x-iy,x+iy]=0 \longrightarrow H_{pJC}=\left(\begin{array}{cc} \theta & a \\ -a^{\dagger} & -\theta \end{array}\right),\ [a,a^{\dagger}]=\mathbf{1}. \tag{32}$$

A comment is in order. In place of the Hamiltonian (27) we can consider the following pseudo Hamiltonian

$$H_p=\omega 1_2\otimes a^{\dagger}a+\frac{\Delta}{2}\tau_3\otimes\mathbf{1}+g\left(\tau_+\otimes a+\tau_-\otimes a^{\dagger}\right) \tag{33}$$

by replacing $\{\sigma_+,\sigma_-,\sigma_3\}$ with $\{\tau_+,\tau_-,\tau_3\}$. This is not hermite (namely, not a convensional one), so we don't know whether this model is useful or not at the current time. It is interesting to note that the model has been considered by [14].

In a forthcoming paper we will extend this "Hamiltonian" and determine its structure in detail like [11].

5 Non–Commutative Hopf and Pseudo Hopf Bundles

In this section we construct a non–commutative version of the Hopf and pseudo Hopf bundles by making (29) and (31) diagonal, which is a "natural" extension in the section 3.

First of all let us recall a Fock space. For a and $a^{\dagger}$ we set $N\equiv a^{\dagger}a$ which is called the number operator, then we have

$$[N,a^{\dagger}]=a^{\dagger},\quad [N,a]=-a,\quad [a^{\dagger},a]=-\mathbf{1}. \tag{34}$$

Let $\mathcal{F}$ be the Fock space generated by $\{a,a^{\dagger},N\}$

$$\mathcal{F}=\mathrm{Vect}_{\mathbf{C}}\{|0\rangle,|1\rangle,\cdots,|n\rangle,\cdots\}. \tag{35}$$

The actions of a and $a^{\dagger}$ on $\mathcal{F}$ are given by

$$a|n\rangle=\sqrt{n}|n-1\rangle,\quad a^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle,\quad N|n\rangle=n|n\rangle \tag{36}$$

where $|0\rangle$ is a normalized vacuum ($a|0\rangle = 0$ and $\langle 0|0\rangle = 1$). From (36) state $|n\rangle$ for $n \geq 1$ are given by

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \ . \tag{37}$$

These states satisfy the orthogonality and completeness conditions

$$\langle m|n\rangle = \delta_{mn} \quad \text{and} \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1}. \tag{38}$$

5.1 Non–Commutative Hopf Bundle

First we make the Hamiltonian (29) diagonal like in Section 2 and research whether "Dirac strings" exist or not in this non–commutative model, which is very interesting from not only quantum optical but also mathematical point of view.

It is easy to see

$$H_{JC} = \begin{pmatrix} \theta & a \\ a^\dagger & -\theta \end{pmatrix} = \begin{pmatrix} 1 & \\ & a^\dagger \frac{1}{\sqrt{N+1}} \end{pmatrix} \begin{pmatrix} \theta & \sqrt{N+1} \\ \sqrt{N+1} & -\theta \end{pmatrix} \begin{pmatrix} 1 & \\ & \frac{1}{\sqrt{N+1}} a \end{pmatrix} \tag{39}$$

from [13]. Then the middle matrix in the right hand side can be considered as a classical one, so we can diagonalize it easily

$$\begin{pmatrix} \theta & \sqrt{N+1} \\ \sqrt{N+1} & -\theta \end{pmatrix} = \begin{cases} A_I \begin{pmatrix} R(N+1) & \\ & -R(N+1) \end{pmatrix} A_I^\dagger \\ A_{II} \begin{pmatrix} R(N+1) & \\ & -R(N+1) \end{pmatrix} A_{II}^\dagger \end{cases} \tag{40}$$

where

$$R(N) = \sqrt{N + \theta^2}$$

and A_I, A_{II} are defined by

$$A_I = \frac{1}{\sqrt{2R(N+1)(R(N+1)+\theta)}} \begin{pmatrix} R(N+1)+\theta & -\sqrt{N+1} \\ \sqrt{N+1} & R(N+1)+\theta \end{pmatrix}, \tag{41}$$

$$A_{II} = \frac{1}{\sqrt{2R(N+1)(R(N+1)-\theta)}} \begin{pmatrix} \sqrt{N+1} & -R(N+1)+\theta \\ R(N+1)-\theta & \sqrt{N+1} \end{pmatrix}. \tag{42}$$

Now let us rewrite (39) by making use of (40) with (41). Inserting the identity

$$\begin{pmatrix} 1 & \\ & \frac{1}{\sqrt{N+1}}a \end{pmatrix}\begin{pmatrix} 1 & \\ & a^\dagger\frac{1}{\sqrt{N+1}} \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

gives

$$\begin{aligned} H_{JC} =& \begin{pmatrix} 1 & \\ & a^\dagger\frac{1}{\sqrt{N+1}} \end{pmatrix} A_I \begin{pmatrix} R(N+1) & \\ & -R(N+1) \end{pmatrix} A_I^\dagger \begin{pmatrix} 1 & \\ & \frac{1}{\sqrt{N+1}}a \end{pmatrix} \\ =& \begin{pmatrix} 1 & \\ & a^\dagger\frac{1}{\sqrt{N+1}} \end{pmatrix} A_I \begin{pmatrix} 1 & \\ & \frac{1}{\sqrt{N+1}}a \end{pmatrix}\begin{pmatrix} 1 & \\ & a^\dagger\frac{1}{\sqrt{N+1}} \end{pmatrix}\begin{pmatrix} R(N+1) & \\ & -R(N+1) \end{pmatrix} \times \\ & \begin{pmatrix} 1 & \\ & \frac{1}{\sqrt{N+1}}a \end{pmatrix}\begin{pmatrix} 1 & \\ & a^\dagger\frac{1}{\sqrt{N+1}} \end{pmatrix} A_I^\dagger \begin{pmatrix} 1 & \\ & \frac{1}{\sqrt{N+1}}a \end{pmatrix} \\ =& U_I \begin{pmatrix} R(N+1) & \\ & -R(N) \end{pmatrix} U_I^\dagger, \end{aligned} \tag{43}$$

where

$$\begin{aligned} U_I =& \begin{pmatrix} \frac{1}{\sqrt{2R(N+1)(R(N+1)+\theta)}} & \\ & \frac{1}{\sqrt{2R(N)(R(N)+\theta)}} \end{pmatrix}\begin{pmatrix} R(N+1)+\theta & -a \\ a^\dagger & R(N)+\theta \end{pmatrix} \\ =& \begin{pmatrix} R(N+1)+\theta & -a \\ a^\dagger & R(N)+\theta \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2R(N+1)(R(N+1)+\theta)}} & \\ & \frac{1}{\sqrt{2R(N)(R(N)+\theta)}} \end{pmatrix}. \end{aligned} \tag{44}$$

Similarly, we can rewrite (39) by making use of (40) with (42). By inserting the identity

$$\begin{pmatrix} \frac{1}{\sqrt{N+1}}a & \\ & 1 \end{pmatrix}\begin{pmatrix} a^\dagger\frac{1}{\sqrt{N+1}} & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

we obtain

$$H_{JC} = U_{II} \begin{pmatrix} R(N) & \\ & -R(N+1) \end{pmatrix} U_{II}^\dagger, \tag{45}$$

where

$$U_{II} = \begin{pmatrix} \frac{1}{\sqrt{2R(N+1)(R(N+1)-\theta)}} & \\ & \frac{1}{\sqrt{2R(N)(R(N)-\theta)}} \end{pmatrix}\begin{pmatrix} a & -R(N+1)+\theta \\ R(N)-\theta & a^\dagger \end{pmatrix}$$

$$= \begin{pmatrix} a & -R(N+1)+\theta \\ R(N)-\theta & a^\dagger \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2R(N)(R(N)-\theta)}} & \\ & \frac{1}{\sqrt{2R(N+1)(R(N+1)-\theta)}} \end{pmatrix}. \qquad (46)$$

Tidying up these we have

$$H_{JC} = \begin{cases} U_I \begin{pmatrix} R(N+1) & \\ & -R(N) \end{pmatrix} U_I^\dagger \\ U_{II} \begin{pmatrix} R(N) & \\ & -R(N+1) \end{pmatrix} U_{II}^\dagger \end{cases} \qquad (47)$$

with U_I and U_{II} above. From the equations

$$R(N+1)|0\rangle = \sqrt{1+\theta^2} > \theta, \quad R(N)|0\rangle = \sqrt{\theta^2} = |\theta|$$

we know

$$(R(N) \pm \theta)\,|0\rangle = (|\theta| \pm \theta)\,|0\rangle,$$

so the strings corresponding to Dirac ones exist in only states $\mathcal{F} \times \{|0\rangle\} \cup \{|0\rangle\} \times \mathcal{F}$ where $\mathcal{F}$ is the Fock space, while in other excited states $\mathcal{F} \times \mathcal{F} \setminus \mathcal{F} \times \{|0\rangle\} \cup \{|0\rangle\} \times \mathcal{F}$ they don't exist [3], see the figure 3. The phenomenon is very interesting. For simplicity we again call these strings Dirac ones in the following.

The "parameter space" of H_{JC} can be identified with $\mathcal{F} \times \mathcal{F} \times \mathbf{R} \ni (*, *, \theta)$, so the domains D_I of U_I and D_{II} of U_{II} are respectively

$$D_I = \mathcal{F} \times \mathcal{F} \times \mathbf{R} - \mathcal{F} \times \{|0\rangle\} \times \mathbf{R}_{\leq 0}, \qquad (48)$$

$$D_{II} = \mathcal{F} \times \mathcal{F} \times \mathbf{R} - (\mathcal{F} \times \{|0\rangle\} \cup \{|0\rangle\} \times \mathcal{F}) \times \mathbf{R}_{\geq 0} \qquad (49)$$

by (44) and (46). We note that

$$D_I \cup D_{II} = \mathcal{F} \times \mathcal{F} \times \mathbf{R} - \mathcal{F} \times \{|0\rangle\} \times \{\theta = 0\}.$$

Then the transition "function" (operator) is given by

$$\Phi_{JC} = \begin{pmatrix} a\frac{1}{\sqrt{N}} & \\ & \frac{1}{\sqrt{N}}a^\dagger \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{N+1}}a & \\ & a^\dagger\frac{1}{\sqrt{N+1}} \end{pmatrix}.$$

[3] We have identified $\mathcal{F} \times \mathcal{F}$ with the space of 2–component vectors over $\mathcal{F}$

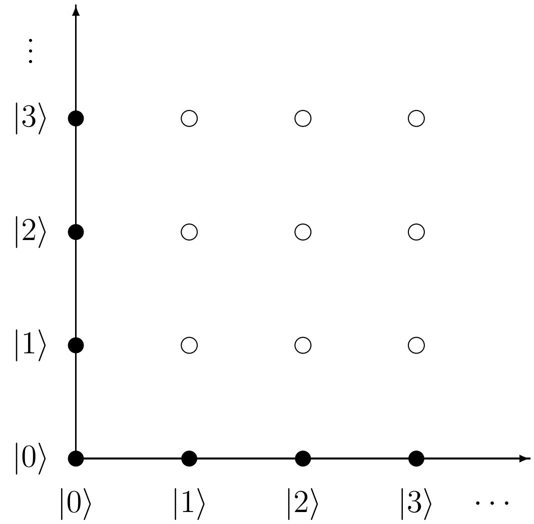

Figure 4: The bases of $\mathcal{F}\times\mathcal{F}$. The black circle means bases giving Dirac strings, while the white one don't.

Therefore the system

$$\{(U_I, D_I), (U_{II}, D_{II}), \Phi_{JC}, D_I \cup D_{II}\} \tag{50}$$

is a non-commutative version of the Hopf bundle (15). The projector in this case becomes

$$\begin{aligned} P_{JC} &= U_I \begin{pmatrix} \mathbf{1} & \\ & \mathbf{0} \end{pmatrix} U_I^\dagger = U_{II} \begin{pmatrix} \mathbf{1} & \\ & \mathbf{0} \end{pmatrix} U_{II}^\dagger \\ &= \begin{cases} \begin{pmatrix} \frac{1}{2R(N+1)} & \\ & \frac{1}{2R(N)} \end{pmatrix} \begin{pmatrix} R(N+1)+\theta & a \\ a^\dagger & R(N)-\theta \end{pmatrix} \\ \begin{pmatrix} R(N+1)+\theta & a \\ a^\dagger & R(N)-\theta \end{pmatrix} \begin{pmatrix} \frac{1}{2R(N+1)} & \\ & \frac{1}{2R(N)} \end{pmatrix} \end{cases}. \end{aligned} \tag{51}$$

Note that the projector P_{JC} is not defined on $\mathcal{F}\times\{|0\rangle\}\times\{\theta=0\} = \mathcal{F}\times\mathcal{F}\times\mathbf{R} - D_I\cup D_{II}$.

A comment is in order. From (51) we obtain a quantum version of (classical) spectral decomposition (a "quantum spectral decomposition" by Suzuki [15])

$$H_{JC} = \begin{pmatrix} R(N+1) & \\ & R(N) \end{pmatrix} P_{JC} - \begin{pmatrix} R(N+1) & \\ & R(N) \end{pmatrix} (\mathbf{1}_2 - P_{JC}). \tag{52}$$

As a bonus of the decomposition let us rederive the calculation of $\mathrm{e}^{-igtH_{JC}}$ which has been given in [10]. The result is

$$\mathrm{e}^{-igtH_{JC}} = \begin{pmatrix} \cos(tgR(N+1)) - i\theta\frac{\sin(tgR(N+1))}{R(N+1)} & -i\frac{\sin(tgR(N+1))}{R(N+1)}a \\ -i\frac{\sin(tgR(N))}{R(N)}a^\dagger & \cos(tgR(N)) + i\theta\frac{\sin(tgR(N))}{R(N)} \end{pmatrix} \tag{53}$$

by making use of (47) (or (52)). We leave it to the readers.

5.2 Non–Commutative Pseudo Hopf Bundle

Similarly, we make the Hamiltonian (31) diagonal like in the preceding subsection to study what a non–commutative version of Dirac strings is.

It is easy to see

$$H_{pJC} = \begin{pmatrix} \theta & a \\ -a^\dagger & -\theta \end{pmatrix} = \begin{pmatrix} 1 & \\ & a^\dagger\frac{1}{\sqrt{N+1}} \end{pmatrix} \begin{pmatrix} \theta & \sqrt{N+1} \\ -\sqrt{N+1} & -\theta \end{pmatrix} \begin{pmatrix} 1 & \\ & \frac{1}{\sqrt{N+1}}a \end{pmatrix}. \tag{54}$$

The middle matrix in the right hand side can be considered as a classical one, so we can diagonalize it easily

$$\begin{pmatrix} \theta & \sqrt{N+1} \\ -\sqrt{N+1} & -\theta \end{pmatrix} = B \begin{pmatrix} S(N+1) & \\ & -S(N+1) \end{pmatrix} B^{-1} \tag{55}$$

where

$$S(N) = \sqrt{\theta^2 - N}$$

and B defined by

$$B = \frac{1}{\sqrt{2S(N+1)(\theta + S(N+1))}} \begin{pmatrix} S(N+1)+\theta & -\sqrt{N+1} \\ -\sqrt{N+1} & S(N+1)+\theta \end{pmatrix}. \tag{56}$$

In this case the situation changes in a drastic manner. Since $S(N+1) = \sqrt{\theta^2 - (N+1)}$ where $N = a^\dagger a$ is the number operator, it is clear that only a restricted subspace of the Fock space $\mathcal{F}$

$$\mathcal{F}_n = \mathrm{Vect}_{\mathbf{C}}\{|0\rangle, |1\rangle, \cdots, |n-1\rangle\}$$

is available if $n < \theta^2 \le n+1$. Moreover, if $0 < \theta^2 \le 1$ there is no subspace that $S(N+1)$ is defined !

Similarly in the preceding subsection we have

$$H_{pJC} = V \begin{pmatrix} S(N+1) & \\ & -S(N) \end{pmatrix} V^{-1} \tag{57}$$

with V defined by

$$\begin{aligned} V &= \begin{pmatrix} \frac{1}{\sqrt{2S(N+1)(S(N+1)+\theta)}} & \\ & \frac{1}{\sqrt{2S(N)(S(N)+\theta)}} \end{pmatrix} \begin{pmatrix} S(N+1)+\theta & -a \\ -a^\dagger & S(N)+\theta \end{pmatrix} \\ &= \begin{pmatrix} R(N+1)+\theta & -a \\ -a^\dagger & R(N)+\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2S(N+1)(S(N+1)+\theta)}} & \\ & \frac{1}{\sqrt{2S(N)(S(N)+\theta)}} \end{pmatrix}. \end{aligned} \tag{58}$$

We note that V above satisfies the relation

$$V^\dagger \mathbf{J} V = \mathbf{J}, \quad \text{where} \quad \mathbf{J} = \begin{pmatrix} \mathbf{1} & \\ & -\mathbf{1} \end{pmatrix} \Longrightarrow V^{-1} = \mathbf{J} V^\dagger \mathbf{J}.$$

The "parameter space" of H_{pJC} can be identified with

$$\bigcup_{n \in \mathbf{N}} \mathcal{F}_n \times \mathcal{F}_{n+1} \times \{\theta \in \mathbf{R}_{>0} \mid n < \theta^2 \le n+1\} \ni (*, *, \theta).$$

The projector in this case becomes

$$\begin{aligned} Q_{pJC} &= V \begin{pmatrix} \mathbf{1} & \\ & \mathbf{0} \end{pmatrix} V^{-1} \\ &= \begin{cases} \begin{pmatrix} \frac{1}{2S(N+1)} & \\ & \frac{1}{2S(N)} \end{pmatrix} \begin{pmatrix} S(N+1)+\theta & a \\ -a^\dagger & S(N)-\theta \end{pmatrix} \\ \begin{pmatrix} S(N+1)+\theta & a \\ -a^\dagger & S(N)-\theta \end{pmatrix} \begin{pmatrix} \frac{1}{2S(N+1)} & \\ & \frac{1}{2S(N)} \end{pmatrix}. \end{cases} \end{aligned} \tag{59}$$

It is easy to see the relations

$$Q_{pJC}^2 = Q_{pJC}, \quad \mathbf{J} Q_{pJC} \mathbf{J} = Q_{pJC}^\dagger.$$

A comment is in order. From (59) we obtain a quantum version of (classical) spectral decomposition

$$H_{pJC} = \begin{pmatrix} S(N+1) & \\ & S(N) \end{pmatrix} Q_{pJC} - \begin{pmatrix} S(N+1) & \\ & S(N) \end{pmatrix} (\mathbf{1}_2 - Q_{pJC}). \tag{60}$$

As a bonus of the decomposition let us rederive the calculation of $\mathrm{e}^{-igtH_{pJC}}$ which seems to be new. The result is

$$\mathrm{e}^{-igtH_{pJC}} = \begin{pmatrix} \cos(tgS(N+1)) - i\theta\frac{\sin(tgS(N+1))}{S(N+1)} & -i\frac{\sin(tgS(N+1))}{S(N+1)}a \\ i\frac{\sin(tgS(N))}{S(N)}a^{\dagger} & \cos(tgS(N)) + i\theta\frac{\sin(tgS(N))}{S(N)} \end{pmatrix} \tag{61}$$

by making use of (57) (or (60)). We leave it to the readers.

We note once more that $\mathrm{e}^{-igtH_{pJC}}$ is not unitary, but satisfies the relation

$$\left(\mathrm{e}^{-igtH_{pJC}}\right)^{\dagger} \mathbf{J} \mathrm{e}^{-igtH_{pJC}} = \mathbf{J}.$$

6 Non–Commutative Veronese and Pseudo Veronese Mappings

In this section we construct a non–commutative version of the (classical) Veronese Mapping and its noncompact counterpart.

6.1 Non–Commutative Veronese Mapping

Let us make a brief review of the Veronese mapping. The map

$$\mathbf{C}P^1 \longrightarrow \mathbf{C}P^n$$

is defined as

$$[z_1 : z_2] \longrightarrow \left[z_1^n : \sqrt{{}_nC_1} z_1^{n-1} z_2 : \cdots : \sqrt{{}_nC_j} z_1^{n-j} z_2^j : \cdots : \sqrt{{}_nC_{n-1}} z_1 z_2^{n-1} : z_2^n\right]$$

by making use of the homogeneous coordinate, see Appendix A. We also have another expression of this map by using

$$S^1_{\mathbf{C}} \longrightarrow S^n_{\mathbf{C}} \ : \ v_1 \equiv \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longrightarrow \ v_n \equiv \begin{pmatrix} z_1^n \\ \sqrt{{}_nC_1} z_1^{n-1} z_2 \\ \vdots \\ \sqrt{{}_nC_j} z_1^{n-j} z_2^j \\ \vdots \\ \sqrt{{}_nC_{n-1}} z_1 z_2^{n-1} \\ z_2^n \end{pmatrix}, \quad |z_1|^2 + |z_2|^2 = 1$$

where $S^m_{\mathbf{C}} = \left\{ (w_1, w_2, \cdots, w_{m+1})^{\mathrm{T}} \in \mathbf{C}^{m+1} \mid \sum_{j=1}^{m+1} |w_j|^2 = 1 \right\} \cong S^{2m+1}$ and $\mathbf{C}P^m = S^m_{\mathbf{C}}/U(1)$. Then the Veronese mapping is also written as

$$\mathbf{C}P^1 \longrightarrow \mathbf{C}P^n \ : \ P_1 = v_1 v_1^\dagger \ \longmapsto \ P_n = v_n v_n^\dagger.$$

by using projectors, which is easy to understand.

Moreover, the local map ($z \equiv z_2/z_1$) is given as

$$\mathbf{C} \longrightarrow \mathbf{C}^n : z \longrightarrow \begin{pmatrix} \sqrt{{}_nC_1} z \\ \vdots \\ \sqrt{{}_nC_j} z^j \\ \vdots \\ \sqrt{{}_nC_{n-1}} z^{n-1} \\ z^n \end{pmatrix}.$$

See the following picture as a whole.

$$
\begin{array}{ccc}
S^1_{\mathbf{C}} & \longrightarrow & S^n_{\mathbf{C}} \\
\downarrow & & \downarrow \\
\mathbf{C}P^1 & \longrightarrow & \mathbf{C}P^n \\
\uparrow & & \uparrow \\
\mathbf{C} & \longrightarrow & \mathbf{C}^n
\end{array}
$$

Next we want to consider a non–commutative version of the map. If we set

$$
\mathcal{A} \equiv \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} \frac{R(N+1)+\theta}{\sqrt{2R(N+1)(R(N+1)+\theta)}} \\ \frac{1}{\sqrt{2R(N)(R(N)+\theta)}} a^\dagger \end{pmatrix} \tag{62}
$$

from U_I in (44), then

$$
\mathcal{A}^\dagger \mathcal{A} = X_0^2 + Y_0^\dagger Y_0 = \mathbf{1} \quad \text{and} \quad Y_0 X_0^{-1} = \frac{1}{R(N)+\theta} a^\dagger \equiv Z.
$$

That is, $\mathcal{A} = (X_0, Y_0)^T$ is a non–commutative sphere and Z is a kind of "stereographic projection" of the sphere. It is easy to see the following

$$
\mathbf{1} + Z^\dagger Z = \frac{2R(N+1)}{R(N+1)+\theta} = X_0^{-2} \Longrightarrow X_0 = \left(\mathbf{1} + Z^\dagger Z\right)^{-1/2}. \tag{63}
$$

Here let us introduce new notations for the following. For $j \geq 0$ we set

$$
X_{-j} = \frac{R(N+1-j)+\theta}{\sqrt{2R(N+1-j)(R(N+1-j)+\theta)}}, \tag{64}
$$

$$
Y_{-j} = \sqrt{\frac{N-j}{N}} \frac{1}{\sqrt{2R(N-j)(R(N-j)+\theta)}} a^\dagger. \tag{65}
$$

We list some useful formulas.

$$
X_{-j}^2 + Y_{-j}^\dagger Y_{-j} = \mathbf{1} \quad \text{and} \quad Y_{-j}^\dagger Y_{-j} = Y_{-j+1} Y_{-j+1}^\dagger \quad \text{for} \quad j \geq 0. \tag{66}
$$

Now we are in a position to define a quantum version of the Veronese mapping which plays a very important role in "classical" Mathematics.

$$\mathcal{A}=\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} \longrightarrow \mathcal{A}_n=\begin{pmatrix} X_0^n \\ \sqrt{{}_nC_1}Y_0X_0^{n-1} \\ \vdots \\ \sqrt{{}_nC_j}Y_{-(j-1)}Y_{-(j-2)}\cdots Y_{-1}Y_0X_0^{n-j} \\ \vdots \\ \sqrt{{}_nC_{n-1}}Y_{-(n-2)}Y_{-(n-3)}\cdots Y_{-1}Y_0X_0 \\ Y_{-(n-1)}Y_{-(n-2)}Y_{-(n-3)}\cdots Y_{-1}Y_0 \end{pmatrix}. \tag{67}$$

Then it is not difficult to see

$$\mathcal{A}_n^\dagger\mathcal{A}_n=\left(X_0^2+Y_0^\dagger Y_0\right)^n=\left(\mathcal{A}^\dagger\mathcal{A}\right)^n=\mathbf{1}.$$

From this we can define the projectors which correspond to projective spaces like

$$\mathcal{P}_n=\mathcal{A}_n\mathcal{A}_n^\dagger, \quad \mathcal{P}_1=\mathcal{A}\mathcal{A}^\dagger, \tag{68}$$

so the map

$$\mathcal{P}_1 \ \longrightarrow \ \mathcal{P}_n \tag{69}$$

is a non-commutative version of the Veronese mapping.

Next, we define a local "coordinate" of the Veronese mapping defined above.

$$\mathcal{A}_n=\begin{pmatrix} \mathbf{1} \\ \sqrt{{}_nC_1}Y_0X_0^{-1} \\ \vdots \\ \sqrt{{}_nC_j}Y_{-(j-1)}Y_{-(j-2)}\cdots Y_{-1}Y_0X_0^{-j} \\ \vdots \\ \sqrt{{}_nC_{n-1}}Y_{-(n-2)}Y_{-(n-3)}\cdots Y_{-1}Y_0X_0^{-(n-1)} \\ Y_{-(n-1)}Y_{-(n-2)}Y_{-(n-3)}\cdots Y_{-1}Y_0X_0^{-n} \end{pmatrix} X_0^n$$

$$=\cdots$$

$$= \begin{pmatrix} \mathbf{1} \\ \sqrt{{}_nC_1}Y_0X_0^{-1} \\ \vdots \\ \sqrt{{}_nC_j}Y_{-(j-1)}X_{-(j-1)}^{-1}Y_{-(j-2)}X_{-(j-2)}^{-1}\cdots Y_{-1}X_{-1}^{-1}Y_0X_0^{-1} \\ \vdots \\ \sqrt{{}_nC_{n-1}}Y_{-(n-2)}X_{-(n-2)}^{-1}Y_{-(n-3)}X_{-(n-3)}^{-1}\cdots Y_{-1}X_{-1}^{-1}Y_0X_0^{-1} \\ Y_{-(n-1)}X_{-(n-1)}^{-1}Y_{-(n-2)}X_{-(n-2)}^{-1}Y_{-(n-3)}X_{-(n-3)}^{-1}\cdots Y_{-1}X_{-1}^{-1}Y_0X_0^{-1} \end{pmatrix} X_0^n$$

where we have used the relation

$$Y_{-j}X_{-k}^{-1} = X_{-(k+1)}^{-1}Y_{-j}$$

due to $a^\dagger$ in Y_{-j}. Moreover, by (64) and (65)

$$Y_{-j}X_{-j}^{-1} = \sqrt{\frac{N-j}{N}}\frac{1}{R(N-j)+\theta}a^\dagger \equiv Z_{-j} \quad \text{for} \quad j \geq 0.$$

Note that $Z_0 = Z$. Therefore by using (63) we have

$$\mathcal{A}_n = \begin{pmatrix} \mathbf{1} \\ \sqrt{{}_nC_1}Z_0 \\ \vdots \\ \sqrt{{}_nC_j}Z_{-(j-1)}Z_{-(j-2)}\cdots Z_{-1}Z_0 \\ \vdots \\ \sqrt{{}_nC_{n-1}}Z_{-(n-2)}Z_{-(n-3)}\cdots Z_{-1}Z_0 \\ Z_{-(n-1)}Z_{-(n-2)}Z_{-(n-3)}\cdots Z_{-1}Z_0 \end{pmatrix} \left(\mathbf{1} + Z_0^\dagger Z_0\right)^{-n/2}. \tag{70}$$

Now if we define

$$\mathcal{Z}_n = \begin{pmatrix} \sqrt{{}_nC_1}Z_0 \\ \vdots \\ \sqrt{{}_nC_j}Z_{-(j-1)}Z_{-(j-2)}\cdots Z_{-1}Z_0 \\ \vdots \\ \sqrt{{}_nC_{n-1}}Z_{-(n-2)}Z_{-(n-3)}\cdots Z_{-1}Z_0 \\ Z_{-(n-1)}Z_{-(n-2)}Z_{-(n-3)}\cdots Z_{-1}Z_0 \end{pmatrix}, \tag{71}$$

then

$$\mathcal{A}_n = \begin{pmatrix} \mathbf{1} \\ \mathcal{Z}_n \end{pmatrix} \left(\mathbf{1} + Z_0^\dagger Z_0\right)^{-n/2}.$$

and it is easy to show

$$\mathbf{1} + \mathcal{Z}_n^\dagger \mathcal{Z}_n = \left(\mathbf{1} + Z_0^\dagger Z_0\right)^n,$$

so we obtain

$$\begin{aligned}
\mathcal{P}_n &= \mathcal{A}_n \mathcal{A}_n^\dagger \\
&= \begin{pmatrix} \mathbf{1} \\ \mathcal{Z}_n \end{pmatrix} \left(\mathbf{1} + \mathcal{Z}_n^\dagger \mathcal{Z}_n\right)^{-1} \left(\mathbf{1},\ \mathcal{Z}_n^\dagger\right) \\
&= \begin{pmatrix} \left(\mathbf{1} + \mathcal{Z}_n^\dagger \mathcal{Z}_n\right)^{-1} & \left(\mathbf{1} + \mathcal{Z}_n^\dagger \mathcal{Z}_n\right)^{-1} \mathcal{Z}_n^\dagger \\ \mathcal{Z}_n \left(\mathbf{1} + \mathcal{Z}_n^\dagger \mathcal{Z}_n\right)^{-1} & \mathcal{Z}_n \left(\mathbf{1} + \mathcal{Z}_n^\dagger \mathcal{Z}_n\right)^{-1} \mathcal{Z}_n^\dagger \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1} & -\mathcal{Z}_n^\dagger \\ \mathcal{Z}_n & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \\ & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{1} & -\mathcal{Z}_n^\dagger \\ \mathcal{Z}_n & \mathbf{1} \end{pmatrix}^{-1}.
\end{aligned} \tag{72}$$

This is the Oike expression in [16], see also Appendix B.

A comment is in order. Two of important properties which the classical Veronese mapping has are

1. The Veronese mapping $\mathbf{C}P^1 \longrightarrow \mathbf{C}P^n$ has the **mapping degree** n
2. The Veronese surface (which is the image of Veronese mapping) is a **minimal surface** in $\mathbf{C}P^n$

Since we have constructed a non–commutative version of the Veronese mapping, a natural question arises : What are non–commutative versions corresponding to 1. and 2. above ? These are very interesting problems from the view point of non–commutative "differential" geometry. It is worth challenging.

6.2 Non–Commutative Pseudo Veronese Mapping

We make a review of the noncompact one of Veronese mapping which we call a pseudo Veronese mapping. First let us define the manifold $\mathbf{C}Q^n$ which is not always well known.

$$\mathbf{C}Q^n = \left\{ Q \in M(n+1;\mathbf{C}) \mid Q^2 = Q,\ J_n Q J_n = Q^\dagger \text{ and } \mathrm{tr}Q = 1 \right\} \tag{73}$$

where J_n is a matrix defined by

$$J_n = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & (-1)^n \end{pmatrix} \quad \in \quad M(n+1;\mathbf{C}).$$

We note that this J_n is not a convensional one. Usually it is taken as

$$\widetilde{J}_n = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & -1 \end{pmatrix} \quad \in \quad M(n+1;\mathbf{C}).$$

For the space $H^n_{\mathbf{C}}$ defined by

$$H^n_{\mathbf{C}} = \left\{ v \in \mathbf{C}^{n+1} \mid v^\dagger J_n v = 1 \right\} \tag{74}$$

we can define a map

$$H^n_{\mathbf{C}} \longrightarrow \mathbf{C}Q^n \ :\ v \longmapsto Q = vv^\dagger J_n. \tag{75}$$

For $Q_1 \in \mathbf{C}Q^1$ it can be written as

$$\begin{aligned} Q_1 &= \begin{pmatrix} \alpha & -\bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \begin{pmatrix} \alpha & -\bar{\beta} \\ -\beta & \bar{\alpha} \end{pmatrix}^{-1}, \quad |\alpha|^2 - |\beta|^2 = 1 \\ &= v_1 v_1^\dagger J_1, \end{aligned}$$

where

$$v_1 = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix} \in H^1_{\mathbf{C}}.$$

For this v_1 we define v_n as

$$v_n = \begin{pmatrix} \alpha^n \\ \sqrt{{}_nC_1}\alpha^{n-1}(-\beta) \\ \vdots \\ \sqrt{{}_nC_j}\alpha^{n-j}(-\beta)^j \\ \vdots \\ (-\beta)^n \end{pmatrix}. \tag{76}$$

Then it is easy to see

$$v_n^\dagger J_n v_n = \left(|\alpha|^2 - |\beta|^2\right)^n = 1,$$

so $v_n \in H^n_{\mathbf{C}}$. Namely, we defined the map

$$H^1_{\mathbf{C}} \longrightarrow H^n_{\mathbf{C}} \ : \ v_1 \longmapsto v_n.$$

Therefore, we have the noncompact one of Veronese mapping

$$\mathbf{C}Q^1 \longrightarrow \mathbf{C}Q^n \ : \ Q_1 = v_1 v_1^\dagger J_1 \longmapsto Q_n = v_n v_n^\dagger J_n. \tag{77}$$

For a tentative terminology let us call this a pseudo Veronese mapping.

Next, let us consider a local coordinate system. From (76)

$$v_n = \begin{pmatrix} 1 \\ \sqrt{{}_nC_1}w \\ \vdots \\ \sqrt{{}_nC_j}w^j \\ \vdots \\ w^n \end{pmatrix} \alpha^n \quad \text{where} \quad w = -\alpha/\beta$$

then it is easy to check $|w|^2 < 1$. We define a domain like (open) hyperbolic pillar

$$D^n_J = \left\{ v \in \mathbf{C}^n \mid v^\dagger J_{n-1} v < 1 \right\}.$$

Then

$$D_J^1 \longrightarrow D_J^n \ : \ w \longmapsto \begin{pmatrix} \sqrt{{}_nC_1} w \\ \vdots \\ \sqrt{{}_nC_j} w^j \\ \vdots \\ w^n \end{pmatrix}$$

is a local map that we are looking for. As a whole see the following picture.

$$\begin{tikzcd} H_{\mathbf{C}}^1 \arrow[r] \arrow[d] & H_{\mathbf{C}}^n \arrow[d] \\ \mathbf{C}Q^1 \arrow[r] & \mathbf{C}Q^n \\ {D_J}^1 \arrow[r] \arrow[u] & {D_J}^n \arrow[u] \end{tikzcd}$$

Next we want to consider a non–commutative version of the map. If we set

$$\mathcal{B} \equiv \begin{pmatrix} \Gamma_0 \\ \Omega_0 \end{pmatrix} = \begin{pmatrix} \frac{S(N+1)+\theta}{\sqrt{2S(N+1)(S(N+1)+\theta)}} \\ -\frac{1}{\sqrt{2S(N)(S(N)+\theta)}} a^{\dagger} \end{pmatrix} \tag{78}$$

from V in (58), then

$$\mathcal{B}^{\dagger}\mathbf{J}\mathcal{B} = \Gamma_0^2 - \Omega_0^{\dagger}\Omega_0 = \mathbf{1} \quad \text{and} \quad \Omega_0\Gamma_0^{-1} = \frac{-1}{S(N)+\theta} a^{\dagger} \equiv W.$$

That is, $\mathcal{B} = (\Gamma_0, \Omega_0)^T$ is a non–commutative hyperboloid and W is a kind of "stereographic projection" of the hyperboloid. It is easy to see the following

$$\mathbf{1} - W^{\dagger}W = \frac{2S(N+1)}{S(N+1)+\theta} = \Gamma_0^{-2} \Longrightarrow \Gamma_0 = \left(\mathbf{1} - W^{\dagger}W\right)^{-1/2}. \tag{79}$$

Here let us introduce new notations for the following. For $j \geq 0$ we set

$$\Gamma_{-j} \ = \ \frac{S(N+1-j)+\theta}{\sqrt{2S(N+1-j)(S(N+1-j)+\theta)}}, \tag{80}$$

$$\Omega_{-j} = -\sqrt{\frac{N-j}{N}} \frac{1}{\sqrt{2S(N-j)(S(N-j)+\theta)}} a^{\dagger}. \tag{81}$$

We list some useful formulas.

$$\Gamma_{-j}^2 - \Omega_{-j}^\dagger \Omega_{-j} = \mathbf{1} \quad \text{and} \quad \Omega_{-j}^\dagger \Omega_{-j} = \Omega_{-j+1} \Omega_{-j+1}^\dagger \quad \text{for} \quad j \geq 0. \tag{82}$$

Now we are in a position to define a non–commutative version of the pseudo Veronese mapping.

$$\mathcal{B} = \begin{pmatrix} \Gamma_0 \\ \Omega_0 \end{pmatrix} \longrightarrow \mathcal{B}_n = \begin{pmatrix} \Gamma_0^n \\ \sqrt{{}_nC_1}\Omega_0\Gamma_0^{n-1} \\ \vdots \\ \sqrt{{}_nC_j}\Omega_{-(j-1)}\Omega_{-(j-2)}\cdots\Omega_{-1}\Omega_0\Gamma_0^{n-j} \\ \vdots \\ \sqrt{{}_nC_{n-1}}\Omega_{-(n-2)}\Omega_{-(n-3)}\cdots\Omega_{-1}\Omega_0\Gamma_0 \\ \Omega_{-(n-1)}\Omega_{-(n-2)}\Omega_{-(n-3)}\cdots\Omega_{-1}\Omega_0 \end{pmatrix}. \tag{83}$$

Then it is not difficult to see

$$\mathcal{B}_n^\dagger \mathbf{J}_n \mathcal{B}_n = \left(\Gamma_0^2 - \Omega_0^\dagger\Omega_0\right)^n = \left(\mathcal{B}^\dagger \mathbf{J} \mathcal{B}\right)^n = \mathbf{1},$$

where $\mathbf{J}_n$ ($\mathbf{J}_1 = \mathbf{J}$) is defined by

$$\mathbf{J}_n = \begin{pmatrix} \mathbf{1} & & & & \\ & -\mathbf{1} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & (-1)^n\mathbf{1} \end{pmatrix} = J_n \otimes \mathbf{1}.$$

From this we can define the projectors which correspond to pseudo projective spaces like

$$\mathcal{Q}_n = \mathcal{B}_n\mathcal{B}_n^\dagger\mathbf{J}_n, \quad \mathcal{Q}_1 = \mathcal{B}\mathcal{B}^\dagger\mathbf{J}, \tag{84}$$

so the map

$$\mathcal{Q}_1 \;\longrightarrow\; \mathcal{Q}_n \tag{85}$$

is a non-commutative version of the pseudo Veronese mapping.

7 Non–Commutative Representation Theory

In this section we construct a map (in the non–commutative models) corresponding to spin j–representation for the compact group $SU(2)$ and noncompact group $SU(1,1)\cdots$ a kind of non–commutative version of classical spin representations $\cdots$.

7.1 Non–Commutative Version of $SU(2)$ Case

The construction of spin j–representation ($j \in \mathbf{Z}_{\geq 0} + 1/2$) is well–known. Let us make a brief review within our necessity. For the vector space

$$\mathcal{H}_J = \mathrm{Vect}_{\mathbf{C}}\left\{\sqrt{{}_{J-1}C_k}z^k \mid k \in \{0, 1, \cdots, J-1\}\right\}$$

where $J = 2j + 1 (\in \mathbf{N})$, the inner product in this space is given by

$$< f|g >= \frac{2J}{2\pi}\int_{\mathbf{C}} \frac{d^2z}{(1+|z|^2)^{J+1}} f(z)\overline{g(z)} = \sum_{k=0}^{J-1} a_k \bar{b}_k$$

for $f(z) = \sum_{k=0}^{J-1} \sqrt{{}_{J-1}C_k} a_k z^k$ and $g(z) = \sum_{k=0}^{J-1} \sqrt{{}_{J-1}C_k} b_k z^k$ in $\mathcal{H}_J$. Here d^2z means $dxdy$ for $z = x + iy$.

For example, for $j = 1/2$, $j = 1$ and $j = 3/2$

$$\mathcal{H}_2 = \mathrm{Vect}_{\mathbf{C}}\{1,\ z\}, \quad \mathcal{H}_3 = \mathrm{Vect}_{\mathbf{C}}\left\{1,\ \sqrt{2}z,\ z^2\right\}, \quad \mathcal{H}_4 = \mathrm{Vect}_{\mathbf{C}}\left\{1,\ \sqrt{3}z,\ \sqrt{3}z^2,\ z^3\right\}.$$

Therefore, we identify $\mathcal{H}_J$ with $\mathbf{C}^J$ by

$$f(z) = \sum_{k=0}^{J-1} \sqrt{{}_{J-1}C_k} a_k z^k \longleftrightarrow (a_0, a_1, \cdots, a_{J-1})^{\mathrm{T}}.$$

For

$$A = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in SU(2) \quad \left(|\alpha|^2 + |\beta|^2 = 1\right)$$

the spin j representation

$$\phi_j : SU(2) \longrightarrow SU(J)$$

is defined as

$$\left(\phi_j(A)f\right)(z) = (\alpha + \beta z)^{J-1} f\left(\frac{-\bar{\beta} + \bar{\alpha}z}{\alpha + \beta z}\right) \tag{86}$$

where $f \in \mathcal{H}_J$. It is easy to obtain $\phi_j(A)$ for $j = 1/2$, $j = 1$ and $j = 3/2$.

Namely, the spin 1/2 representation is

$$\phi_{1/2}(A) = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} = A, \tag{87}$$

the spin 1 representation is

$$\phi_1(A) = \begin{pmatrix} \alpha^2 & -\sqrt{2}\alpha\bar{\beta} & \bar{\beta}^2 \\ \sqrt{2}\alpha\beta & |\alpha|^2 - |\beta|^2 & -\sqrt{2}\bar{\alpha}\bar{\beta} \\ \beta^2 & \sqrt{2}\bar{\alpha}\beta & \bar{\alpha}^2 \end{pmatrix}, \tag{88}$$

and the spin 3/2 representation is

$$\phi_{3/2}(A) = \begin{pmatrix} \alpha^3 & -\sqrt{3}\alpha^2\bar{\beta} & \sqrt{3}\alpha\bar{\beta}^2 & -\bar{\beta}^3 \\ \sqrt{3}\alpha^2\beta & (|\alpha|^2 - 2|\beta|^2)\alpha & -(2|\alpha|^2 - |\beta|^2)\bar{\beta} & \sqrt{3}\bar{\alpha}\bar{\beta}^2 \\ \sqrt{3}\alpha\beta^2 & (2|\alpha|^2 - |\beta|^2)\beta & (|\alpha|^2 - 2|\beta|^2)\bar{\alpha} & -\sqrt{3}\bar{\alpha}^2\bar{\beta} \\ \beta^3 & \sqrt{3}\bar{\alpha}\beta^2 & \sqrt{3}\bar{\alpha}^2\beta & \bar{\alpha}^3 \end{pmatrix}. \tag{89}$$

Next we want to consider a non–commutative version of the spin representation. However, since such a theory has not been known as far as we know we must look for mappings corresponding to $\phi_1(A)$ and $\phi_{3/2}(A)$ by (many) trial and error, see Appendix C.

If we set

$$U \equiv U_I = \begin{pmatrix} X_0 & -Y_0^\dagger \\ Y_0 & X_{-1} \end{pmatrix} \ : \ \text{unitary}$$

from (44), then the corresponding map for $\phi_1(A)$ is

$$\Phi_1(U) = \begin{pmatrix} X_0^2 & -\sqrt{2}X_0Y_0^\dagger & Y_0^\dagger Y_{-1}^\dagger \\ \sqrt{2}Y_0X_0 & X_{-1}^2 - Y_{-1}^\dagger Y_{-1} & -\sqrt{2}X_{-1}Y_{-1}^\dagger \\ Y_{-1}Y_0 & \sqrt{2}Y_{-1}X_{-1} & X_{-2}^2 \end{pmatrix} \tag{90}$$

and the corresponding map for $\phi_{3/2}(A)$ is

$$\Phi_{3/2}(U) = \begin{pmatrix} X_0^3 & -\sqrt{3}X_0^2Y_0^\dagger & \sqrt{3}X_0Y_0^\dagger Y_{-1}^\dagger & -Y_0^\dagger Y_{-1}^\dagger Y_{-2}^\dagger \\ \sqrt{3}Y_0X_0^2 & X_{-1}\left(X_{-1}^2 - 2Y_{-1}^\dagger Y_{-1}\right) & -\left(2X_{-1}^2 - Y_{-1}^\dagger Y_{-1}\right)Y_{-1}^\dagger & \sqrt{3}X_{-1}Y_{-1}^\dagger Y_{-2}^\dagger \\ \sqrt{3}Y_{-1}Y_0X_0 & Y_{-1}\left(2X_{-1}^2 - Y_{-1}^\dagger Y_{-1}\right) & X_{-2}\left(X_{-2}^2 - 2Y_{-2}^\dagger Y_{-2}\right) & -\sqrt{3}X_{-2}^2Y_{-2}^\dagger \\ Y_{-2}Y_{-1}Y_0 & \sqrt{3}Y_{-2}Y_{-1}X_{-1} & \sqrt{3}Y_{-2}X_{-2}^2 & X_{-3}^3 \end{pmatrix}.$$

(91)

To check the unitarity of $\Phi_1(U)$ and $\Phi_{3/2}(U)$ is long but straightforward.

For $j \geq 2$ we could not find a general method like (86) which determines $\Phi_j(U)$. However, we know only that the first column of $\Phi_j(U)$ is just $\mathcal{A}_{2j}$ in (67).,

$$\Phi_j(U) = (\mathcal{A}_{2j}, *, \cdots, *) \ : \ \text{unitary}$$

and

$$\Phi_j(U)\begin{pmatrix} \mathbf{1} & & & & \\ & \mathbf{0} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \mathbf{0} \end{pmatrix}\Phi_j(U)^{\dagger} = \mathcal{A}_{2j}\mathcal{A}_{2j}^{\dagger} = \mathcal{P}_{2j}.$$

We leave finding a general method to the readers as a challenging problem.

7.2 Non–Commutative Version of $SU(1,1)$ Case

Let us review some aspects of the theory of unitary representation of $SU(1,1)$ within our necessity.

Let $H^2 \equiv H^2(D)$ be the second Hardy class where D is the open unit disk in $\mathbf{C}$. We consider the spin j representation of the non–compact group $SU(1,1)$. The inner product is defined as

$$< f|g >= \frac{2(2j-1)}{2\pi}\int_D d^2z(1-|z|^2)^{2j-2}f(z)\overline{g(z)} = \sum_{n=0}^{\infty}\frac{n!}{(2j)_n}a_n\bar{b}_n$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in H^2. For $j = \frac{1}{2}$ we must take some renormalization into consideration (we omit it here). Then $\{H^2, <\,|\,>\}$ becomes a complex Hibert space.

Therefore, it is better for us to consider the vector space

$$H_{2j}^2 = \mathrm{Vect}_{\mathbf{C}}\left\{1, \sqrt{2j}z, \cdots, \sqrt{\frac{(2j)_k}{k!}}z^k, \cdots\right\}$$

and the correspondence between H^2_{2j} and $\ell^2(\mathbf{C})$ is given by

$$f(z)=\sum_{n=0}^{\infty}\sqrt{\frac{(2j)_n}{n!}}a_n z^n \ \longleftrightarrow\ (a_0,a_1,\cdots,a_n,\cdots)^{\mathrm{T}},$$

so we identify H^2_{2j} with $\ell^2(\mathbf{C})$ by this correspondence.

For

$$B=\begin{pmatrix}\alpha & -\bar{\beta}\\ -\beta & \bar{\alpha}\end{pmatrix}\in SU(1,1)\quad\left(|\alpha|^2-|\beta|^2=1\right)$$

the spin j unitary representation

$$\psi_j : SU(1,1)\longrightarrow U(\ell^2(\mathbf{C}))$$

is defined as

$$(\psi_j(B)f)(z)=(\alpha+\beta z)^{-2j}f\left(\frac{\bar{\beta}+\bar{\alpha}z}{\alpha+\beta z}\right)\tag{92}$$

where $f\in H^2_{2j}$.

For example, when $f=1$ (a constant) it is easy to see

$$\begin{aligned}(\psi_j(B)1)(z)&=(\alpha+\beta z)^{-2j}\\ &=\frac{1}{\alpha^{2j}}-2j\frac{\beta}{\alpha^{2j+1}}z+\cdots+(-1)^n\frac{(2j)_n}{n!}\frac{\beta^n}{\alpha^{2j+n}}z^n+\cdots\\ &=\frac{1}{\alpha^{2j}}-\sqrt{2j}\frac{\beta}{\alpha^{2j+1}}\sqrt{2j}z+\cdots+(-1)^n\sqrt{\frac{(2j)_n}{n!}}\frac{\beta^n}{\alpha^{2j+n}}\sqrt{\frac{(2j)_n}{n!}}z^n+\cdots\end{aligned}$$

where $(a)_n$ is the Pochammer notation defined by

$$(a)_n=a(a+1)\cdots(a+n-1).$$

Therefore

$$\psi_j(B)1=\begin{pmatrix}\frac{1}{\alpha^{2j}}\\ -\sqrt{2j}\frac{\beta}{\alpha^{2j+1}}\\ \vdots\\ (-1)^n\sqrt{\frac{(2j)_n}{n!}}\frac{\beta^n}{\alpha^{2j+n}}\\ \vdots\end{pmatrix}=\begin{pmatrix}\alpha^{-2j}\\ -\sqrt{2j}\beta\alpha^{-(2j+1)}\\ \vdots\\ (-1)^n\sqrt{\frac{(2j)_n}{n!}}\beta^n\alpha^{-(2j+n)}\\ \vdots\end{pmatrix}.\tag{93}$$

More generally, for $f_k(z)=\sqrt{\frac{(2j)_k}{k!}}z^k$

$$
\begin{aligned}
\left(\psi_j(B)f_k\right)(z) &= \sqrt{\frac{(2j)_k}{k!}}(\bar{\beta}+\bar{\alpha}z)^k(\alpha+\beta z)^{-(2j+k)} \\
&= \sqrt{\frac{(2j)_k}{k!}}\sum_{l=0}^{k}\sum_{n=0}^{\infty}{}_kC_l(-1)^n\frac{(2j+k)_n}{n!}\frac{\beta^n\bar{\beta}^{k-l}\bar{\alpha}^l}{\alpha^{2j+n+k}}z^{l+n},
\end{aligned}
$$

so

$$
\psi_j(B)f_k=\left(\begin{array}{c}
\sqrt{\frac{(2j)_k}{k!}}\frac{\bar{\beta}^k}{\alpha^{2j+k}} \\
\sqrt{\frac{(2j)_k}{k!}}\sqrt{\frac{1}{2j}}\left\{k|\alpha|^2-(2j+k)|\beta|^2\right\}\frac{\bar{\beta}^{k-1}}{\alpha^{2j+k+1}} \\
\sqrt{\frac{(2j)_k}{k!}}\sqrt{\frac{2!}{(2j)_2}}\left\{\frac{k(k-1)}{2}|\alpha|^4-k(2j+k)|\alpha|^2|\beta|^2+\frac{(2j+k)(2j+k+1)}{2}|\beta|^4\right\}\frac{\bar{\beta}^{k-2}}{\alpha^{2j+k+2}} \\
\vdots \\
\vdots
\end{array}\right).
$$

Therefore the matrix defined by

$$
\psi_j(B)=\left(\psi_j(B)f_0,\psi_j(B)f_1,\cdots,\psi_j(B)f_k,\cdots\right) \tag{94}
$$

is the unitary representation that we are looking for.

We set

$$
V=\begin{pmatrix}\Gamma_0 & \Omega_0^{\dagger} \\ \Omega_0 & \Gamma_{-1}\end{pmatrix}\ :\ \text{pseudo unitary } (V^{\dagger}\mathbf{J}V=\mathbf{J})
$$

from (58). In this case it is almost impossible to obtain the explicit unitary operator $\Psi_j(V)$ corresponding to $\psi_j(B)$.

However, we can at least determine the first column of $\Psi_j(V)$ by making use of (93) :

$$
\widehat{\mathcal{B}}\equiv\left(\begin{array}{c}
\Gamma_0^{-2j} \\
\sqrt{2j}\,\Omega_0\Gamma_0^{-(2j+1)} \\
\vdots \\
\sqrt{\frac{(2j)_n}{n!}}\Omega_{-(n-1)}\Omega_{-(n-2)}\Omega_{-(n-3)}\cdots\Omega_{-1}\Omega_0\Gamma_0^{-(2j+n)} \\
\vdots
\end{array}\right) \tag{95}
$$

with Γ_0 and Ω_{-j} in (80) and (81). Then it is not difficult to see

$$
\widehat{\mathcal{B}}^{\dagger}\widehat{\mathcal{B}}=\left(\Gamma_0^2-\Omega_0^{\dagger}\Omega_0\right)^{-2j}=\mathbf{1}.
$$

Therefore, the map making use of projectors

$$\mathcal{Q}_1 = \mathcal{B}\mathcal{B}^\dagger \mathbf{J} \quad \longrightarrow \quad \widehat{\mathcal{P}} = \widehat{\mathcal{B}}\widehat{\mathcal{B}}^\dagger \tag{96}$$

is a non–commutative version of the unitary expression of pseudo Veronese mapping. Compare the discussion here with the one after the equation (83).

We note that if the unitary operator

$$\Psi_j(V) = \left(\widehat{\mathcal{B}}_0, \widehat{\mathcal{B}}_1, \cdots, \widehat{\mathcal{B}}_n, \cdots\right), \quad \widehat{\mathcal{B}}_0 = \widehat{\mathcal{B}}$$

could be defined (we cannot determine $\widehat{\mathcal{B}}_n$ for $n \geq 1$), then we have

$$\Psi_j(V) \begin{pmatrix} \mathbf{1} & & & \\ & \mathbf{0} & & \\ & & \mathbf{0} & \\ & & & \ddots \end{pmatrix} \Psi_j(V)^\dagger = \widehat{\mathcal{B}}\widehat{\mathcal{B}}^\dagger = \widehat{\mathcal{P}}.$$

A comment is in order. In the construction of $\widehat{\mathcal{B}}_n$ we need an infinite number of operators, which means a kind of **second non–commutativization**.

8 Discussion

In this paper we derived a non–commutative version of the Berry model (based on $SU(2)$) arising from the Jaynes–Cummings model in quantum optics and the pseudo Berry model (based on $SU(1,1)$) by changing the generators, and constructed a non–commutative version of the Hopf and pseudo Hopf bundles in the classical case.

The bundle has a kind of Dirac strings in the case of non–commutative Berry model. However, they appear in only states containing the ground one ($\mathcal{F} \times \{|0\rangle\} \cup \{|0\rangle\} \times \mathcal{F} \subset \mathcal{F} \times \mathcal{F}$) and don't appear in excited states, which is very interesting.

In general, a non-commutative version of classical field theory is of course not unique. If our model is a "correct" one, then this paper give an example that classical singularities like Dirac strings are not universal in some non–commutative model. As to general case with higher spins which are not easy, see [15].

Moreover, for the two models a non–commutative version of the Veronese mapping or pseudo Veronese mapping was constructed, and unitary mappings corresponding to (classical) spin representations were constructed though they are not necessarily enough.

The results or methods in the paper will become a starting point to construct a fruitful non–commutative geometry or representation theory.

Last, we would like to make a comment. To develop a "quantum" mathematics we need a rigorous method to treat an analysis or a geometry on infinite dimensional spaces like Fock space. In quantum field theories physicists have given some (interesting) methods, while they are more or less formal from the mathematical point of view. It is a rigorous method which we need. As a trial [19] is recommended.

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Appendix

A Classical Theory of Projective Spaces

Complex projective spaces are typical examples of symmetric spaces and are very tractable, so they are used to construct several examples in both physics and mathematics. We make a review of complex projective spaces within our necessity, see for example [2], [16], [18].

For $n \in \mathbf{N}$ the complex projective space $\mathbf{C}P^n$ is defined as follows : For $\boldsymbol{\zeta}$, $\boldsymbol{\mu} \in \mathbf{C}^{n+1} - \{\mathbf{0}\}$ $\boldsymbol{\zeta}$ is equivalent to $\boldsymbol{\mu}$ ($\boldsymbol{\zeta} \sim \boldsymbol{\mu}$) if and only if $\boldsymbol{\zeta} = \lambda\boldsymbol{\mu}$ for $\lambda \in \mathbf{C} - \{0\}$. We show the equivalent relation class as $[\boldsymbol{\zeta}]$ and set $\mathbf{C}P^n \equiv \mathbf{C}^{n+1} - \{\mathbf{0}\}/\sim$. For $\boldsymbol{\zeta} = (\zeta_0, \zeta_1, \cdots, \zeta_n)$ we write usually as $[\boldsymbol{\zeta}] = [\zeta_0 : \zeta_1 : \cdots : \zeta_n]$. Then it is well–known that $\mathbf{C}P^n$ has $n+1$ local charts, namely

$$\mathbf{C}P^n = \bigcup_{j=0}^{n} U_j \ , \quad U_j = \{[\zeta_0 : \cdots : \zeta_j : \cdots : \zeta_n] \mid \zeta_j \neq 0\}. \tag{97}$$

Since

$$(\zeta_0, \cdots, \zeta_j, \cdots, \zeta_n) = \zeta_j \left(\frac{\zeta_0}{\zeta_j}, \cdots, \frac{\zeta_{j-1}}{\zeta_j}, 1, \frac{\zeta_{j+1}}{\zeta_j}, \cdots, \frac{\zeta_n}{\zeta_j} \right),$$

we have the local coordinate on U_j

$$\left(\frac{\zeta_0}{\zeta_j}, \cdots, \frac{\zeta_{j-1}}{\zeta_j}, \frac{\zeta_{j+1}}{\zeta_j}, \cdots, \frac{\zeta_n}{\zeta_j} \right). \tag{98}$$

However the above definition of $\mathbf{C}P^n$ is not tractable, so we use the well–known expression by projections (see [16])

$$\mathbf{C}P^n \cong G_{1,n+1}(\mathbf{C}) = \{ P \in M(n+1;\mathbf{C}) \mid P^2 = P,\ P = P^\dagger \text{ and } \mathrm{tr}P = 1 \} \tag{99}$$

and the correspondence

$$[\zeta_0 : \zeta_1 : \cdots : \zeta_n] \Longleftrightarrow \frac{1}{|\zeta_0|^2 + |\zeta_1|^2 + \cdots + |\zeta_n|^2} \begin{pmatrix} |\zeta_0|^2 & \zeta_0\bar{\zeta}_1 & \cdot & \cdot & \zeta_0\bar{\zeta}_n \\ \zeta_1\bar{\zeta}_0 & |\zeta_1|^2 & \cdot & \cdot & \zeta_1\bar{\zeta}_n \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \zeta_n\bar{\zeta}_0 & \zeta_n\bar{\zeta}_1 & \cdot & \cdot & |\zeta_n|^2 \end{pmatrix} \equiv P\ . \tag{100}$$

If we set

$$|\boldsymbol{\zeta}\rangle = \frac{1}{\sqrt{\sum_{j=0}^{n} |\zeta_j|^2}} \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \cdot \\ \cdot \\ \zeta_n \end{pmatrix}, \tag{101}$$

then we can write the right hand side of (100) as

$$P = |\boldsymbol{\zeta}\rangle\langle\boldsymbol{\zeta}| \quad \text{and} \quad \langle\boldsymbol{\zeta}|\boldsymbol{\zeta}\rangle = 1. \tag{102}$$

For example on U_0

$$(z_1, z_2, \cdots, z_n) = \left(\frac{\zeta_1}{\zeta_0}, \frac{\zeta_2}{\zeta_0}, \cdots, \frac{\zeta_n}{\zeta_0} \right),$$

we have

$$
\begin{aligned}
P(z_1,\cdots,z_n) &= \frac{1}{1+\sum_{j=1}^{n}|z_j|^2}\begin{pmatrix} 1 & \bar{z}_1 & \cdot\ \cdot & \bar{z}_n \\ z_1 & |z_1|^2 & \cdot\ \cdot & z_1\bar{z}_n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ z_n & z_n\bar{z}_1 & \cdot\ \cdot & |z_n|^2 \end{pmatrix} \\
&= |(z_1,z_2,\cdots,z_n)\rangle\langle(z_1,z_2,\cdots,z_n)|\ ,
\end{aligned}
\tag{103}
$$

where

$$
|(z_1,z_2,\cdots,z_n)\rangle = \frac{1}{\sqrt{1+\sum_{j=1}^{n}|z_j|^2}}\begin{pmatrix} 1 \\ z_1 \\ \cdot \\ \cdot \\ z_n \end{pmatrix}. \tag{104}
$$

To be clearer, let us give a detailed description for the case of $n=1$ and 2.

(a) $n=1$:

$$
\begin{aligned}
P(z) &= \frac{1}{1+|z|^2}\begin{pmatrix} 1 & \bar{z} \\ z & |z|^2 \end{pmatrix} = |z\rangle\langle z|, \\
&\quad \text{where } |z\rangle = \frac{1}{\sqrt{1+|z|^2}}\begin{pmatrix} 1 \\ z \end{pmatrix}, \quad z=\frac{\zeta_1}{\zeta_0}, \quad \text{on } U_0\ ,
\end{aligned}
\tag{105}
$$

$$
\begin{aligned}
P(w) &= \frac{1}{|w|^2+1}\begin{pmatrix} |w|^2 & w \\ \bar{w} & 1 \end{pmatrix} = |w\rangle\langle w|, \\
&\quad \text{where } |w\rangle = \frac{1}{\sqrt{|w|^2+1}}\begin{pmatrix} w \\ 1 \end{pmatrix}, \quad w=\frac{\zeta_0}{\zeta_1}, \quad \text{on } U_1\ .
\end{aligned}
\tag{106}
$$

(b) $n=2$:

$$
P(z_1,z_2) = \frac{1}{1+|z_1|^2+|z_2|^2}\begin{pmatrix} 1 & \bar{z}_1 & \bar{z}_2 \\ z_1 & |z_1|^2 & z_1\bar{z}_2 \\ z_2 & z_2\bar{z}_1 & |z_2|^2 \end{pmatrix} = |(z_1,z_2)\rangle\langle(z_1,z_2)|,
$$

$$\text{where}\quad |(z_1,z_2)\rangle = \frac{1}{\sqrt{1+|z_1|^2+|z_2|^2}}\begin{pmatrix} 1 \\ z_1 \\ z_2 \end{pmatrix}, \quad (z_1,z_2) = \left(\frac{\zeta_1}{\zeta_0},\frac{\zeta_2}{\zeta_0}\right) \quad \text{on } U_0\ , \tag{107}$$

$$P(w_1,w_2) = \frac{1}{|w_1|^2+1+|w_2|^2}\begin{pmatrix} |w_1|^2 & w_1 & w_1\bar{w}_2 \\ \bar{w}_1 & 1 & \bar{w}_2 \\ w_2\bar{w}_1 & w_2 & |w_2|^2 \end{pmatrix} = |(w_1,w_2)\rangle\langle(w_1,w_2)|,$$

$$\text{where}\quad |(w_1,w_2)\rangle = \frac{1}{\sqrt{|w_1|^2+1+|w_2|^2}}\begin{pmatrix} w_1 \\ 1 \\ w_2 \end{pmatrix}, \quad (w_1,w_2) = \left(\frac{\zeta_0}{\zeta_1},\frac{\zeta_2}{\zeta_1}\right) \quad \text{on } U_1\ , \tag{108}$$

$$P(v_1,v_2) = \frac{1}{|v_1|^2+|v_2|^2+1}\begin{pmatrix} |v_1|^2 & v_1\bar{v}_2 & v_1 \\ v_2\bar{v}_1 & |v_2|^2 & v_2 \\ \bar{v}_1 & \bar{v}_2 & 1 \end{pmatrix} = |(v_1,v_2)\rangle\langle(v_1,v_2)|,$$

$$\text{where}\quad |(v_1,v_2)\rangle = \frac{1}{\sqrt{|v_1|^2+|v_2|^2+1}}\begin{pmatrix} v_1 \\ v_2 \\ 1 \end{pmatrix}, \quad (v_1,v_2) = \left(\frac{\zeta_0}{\zeta_2},\frac{\zeta_1}{\zeta_2}\right) \quad \text{on } U_2\ . \tag{109}$$

B Local Coordinate of the Projector

We give a proof to the last formula in (72).

By making use of the expression by Oike in [16] (we don't repeat it here)

$$\mathcal{P}(\mathcal{Z}) = \begin{pmatrix} \mathbf{1} & -\mathcal{Z}^\dagger \\ \mathcal{Z} & \mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{1} & \\ & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{1} & -\mathcal{Z}^\dagger \\ \mathcal{Z} & \mathbf{1} \end{pmatrix}^{-1} \tag{110}$$

where $\mathcal{Z}$ is some operator on the Fock space $\mathcal{F}$. Let us rewrite this into more useful form. From the simple relation

$$\begin{pmatrix} \mathbf{1} & \mathcal{Z}^\dagger \\ -\mathcal{Z} & \mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{1} & -\mathcal{Z}^\dagger \\ \mathcal{Z} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1}+\mathcal{Z}^\dagger\mathcal{Z} & \\ & \mathbf{1}+\mathcal{Z}\mathcal{Z}^\dagger \end{pmatrix}$$

we have

$$\begin{pmatrix} \mathbf{1} & -\mathcal{Z}^\dagger \\ \mathcal{Z} & \mathbf{1} \end{pmatrix}^{-1} = \begin{pmatrix} (\mathbf{1}+\mathcal{Z}^\dagger\mathcal{Z})^{-1} & \\ & (\mathbf{1}+\mathcal{Z}\mathcal{Z}^\dagger)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathcal{Z}^\dagger \\ -\mathcal{Z} & \mathbf{1} \end{pmatrix}.$$

Inserting this into (110) and some calculation leads to

$$\mathcal{P}(\mathcal{Z}) = \begin{pmatrix} (\mathbf{1}+\mathcal{Z}^\dagger\mathcal{Z})^{-1} & (\mathbf{1}+\mathcal{Z}^\dagger\mathcal{Z})^{-1}\mathcal{Z}^\dagger \\ \mathcal{Z}(\mathbf{1}+\mathcal{Z}^\dagger\mathcal{Z})^{-1} & \mathcal{Z}(\mathbf{1}+\mathcal{Z}^\dagger\mathcal{Z})^{-1}\mathcal{Z}^\dagger \end{pmatrix}. \tag{111}$$

Comparing (111) with (51) we obtain the "local coordinate"

$$\mathcal{Z} = \frac{1}{R(N)+\theta}a^\dagger = a^\dagger\frac{1}{R(N+1)+\theta} \tag{112}$$

where $R(N) = \sqrt{N+\theta^2}$. $\mathcal{Z}$ obtained by "stereographic projection" is a kind of complex coordinate.

Now if we take a classical limit $a \longrightarrow x - iy$, $a^\dagger \longrightarrow x + iy$ and $\theta = z$ then

$$Z_c = \frac{x+iy}{r+z} \tag{113}$$

where $r = \sqrt{x^2+y^2+z^2}$. This is nothing but a well–known one for (16).

C Some Calculations of First Chern Class

We calculate the first Chern class of some vector bundles on $\mathbf{C}P^1$ and show that the mapping degree of Veronese mapping is just n.

We write our definition of $\mathbf{C}P^n$ once more :

$$\mathbf{C}P^n = \{P \in M(n+1;\mathbf{C}) \mid P^2 = P,\ P = P^\dagger \text{ and } \mathrm{tr}P = 1\}.$$

On this space we define a canonical vector bundle like

$$E_n = \left\{(P,v) \in \mathbf{C}P^n \times \mathbf{C}^{n+1} \mid Pv = v\right\},$$
$$\pi\ :\ E_n \longrightarrow \mathbf{C}P^n, \quad \pi(P,v) = P.$$

Then the system $\xi_n = \{\mathbf{C}, E_n, \pi, \mathbf{C}P^n\}$ is called the canonical line bundle (because P is rank one), see [2], [16]. This is one of most important vector bundles.

Let us calculate the first Chern class of ξ_1. For the local coordinate z in section 6.1, P can be written as

$$P(z) = \frac{1}{1+|z|^2}\begin{pmatrix} 1 & \bar{z} \\ z & |z|^2 \end{pmatrix}, \quad v(z) = \alpha \begin{pmatrix} 1 \\ z \end{pmatrix} \quad (\alpha \in \mathbf{C}). \tag{114}$$

Then the canonical connection $\mathcal{A}$ and its curvature $\mathcal{F}$ can be written as

$$\mathcal{A} = \frac{\bar{z}}{1+|z|^2}dz, \quad \mathcal{F} = d\mathcal{A} = \frac{1}{(1+|z|^2)^2}d\bar{z}\wedge dz. \tag{115}$$

Let χ be the Veronese mapping in section 6.1 ($\chi : \mathbf{C}P^1 \longrightarrow \mathbf{C}P^n$). then we can consider the pull–back bundle $\chi^*\xi_n = \{\mathbf{C}, \chi^*(E_n), \pi, \mathbf{C}P^1\}$ where

$$\chi^*(E_n) = \left\{(P,v) \in \mathbf{C}P^1 \times \mathbf{C}^{n+1} \mid \chi(P)v = v\right\}$$
$$\pi : \chi^*(E_n) \longrightarrow \mathbf{C}P^1, \quad \pi(P,v) = P.$$

See the following picture.

$$\begin{tikzcd} \chi^*(E_n) \arrow[r, dashed] \arrow[d] & E_n \arrow[d] \\ \mathbf{C}P^1 \arrow[r, "\chi"] & \mathbf{C}P^n \end{tikzcd}$$

Let us give a local description. For z in (114)

$$\chi(P(z)) = \frac{1}{(1+|z|^2)^n}\begin{pmatrix} 1 & \psi(z)^\dagger \\ \psi(z) & \psi(z)\psi(z)^\dagger \end{pmatrix}, \quad v(z) = \alpha \begin{pmatrix} 1 \\ \psi(z) \end{pmatrix} \quad (\alpha \in \mathbf{C})$$

where $\psi(z)$ is the map defined in section 6.1

$$\psi(z) = \begin{pmatrix} \sqrt{{}_nC_1}z \\ \vdots \\ \sqrt{{}_nC_j}z^j \\ \vdots \\ \sqrt{{}_nC_{n-1}}z^{n-1} \\ z^n \end{pmatrix} \quad \Longrightarrow \quad 1 + \psi(z)^\dagger\psi(z) = (1+|z|^2)^n.$$

Now the connection and curvature of the pull–backed bundle are given by

$$\mathcal{A}_n = (1+\psi(z)^\dagger\psi(z))^{-1}\psi(z)^\dagger d\psi(z), \quad \mathcal{F}_n = d\mathcal{A}_n. \tag{116}$$

Let us calculate : it is easy to see

$$\begin{aligned}
\mathcal{A}_n &= \frac{{}_nC_1 + \cdots + j\,{}_nC_j|z|^{2(j-1)} + \cdots + n\,{}_nC_n|z|^{2(n-1)}}{(1+|z|^2)^n}\bar{z}dz \\
&= \frac{\frac{d}{d(|z|^2)}\left({}_nC_1|z|^2 + + \cdots + {}_nC_j|z|^{2j} + \cdots + {}_nC_n|z|^{2n}\right)}{(1+|z|^2)^n}\bar{z}dz \\
&= \frac{\frac{d}{d(|z|^2)}\left((1+|z|^2)^n - 1\right)}{(1+|z|^2)^n}\bar{z}dz \\
&= \frac{n(1+|z|^2)^{n-1}}{(1+|z|^2)^n}\bar{z}dz \\
&= n\frac{\bar{z}}{1+|z|^2}dz \\
&= n\mathcal{A},
\end{aligned}$$

therefore

$$\mathcal{F}_n = n\mathcal{F} = n\frac{1}{(1+|z|^2)^2}d\bar{z}\wedge dz.$$

As a result we have

$$\mathrm{Ch}_1(\chi^*\xi_n) = \frac{1}{2\pi i}\int_{\mathbf{C}} n\frac{1}{(1+|z|^2)^2}d\bar{z}\wedge dz = n. \tag{117}$$

As to calculations of geometric objects like Chern classes or holonomies on quantum computation see for example [16] or [17].

D Difficulty of Tensor Decomposition

We point out a difficulty in obtaining the formula (90) or (91) by decomposing tensor products of V.

To obtain the formula (88) there is another method which uses a decomposition of the tensor product $A \otimes A$. Let us introduce. For

$$A = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in SU(2)$$

we have

$$A \otimes A = \begin{pmatrix} \alpha^2 & -\alpha\bar{\beta} & -\alpha\bar{\beta} & \bar{\beta}^2 \\ \alpha\beta & |\alpha|^2 & -|\beta|^2 & -\bar{\alpha}\bar{\beta} \\ \alpha\beta & -|\beta|^2 & |\alpha|^2 & -\bar{\alpha}\bar{\beta} \\ \beta^2 & \bar{\alpha}\beta & \bar{\alpha}\beta & \bar{\alpha}^2 \end{pmatrix}.$$

For the matrix T coming from the Clebsch–Gordan decomposition

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

it is easy to see

$$T^\dagger (A \otimes A) T = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & & & \\ & \alpha^2 & -\sqrt{2}\alpha\bar{\beta} & \bar{\beta}^2 \\ & \sqrt{2}\alpha\beta & |\alpha|^2 - |\beta|^2 & -\sqrt{2}\bar{\alpha}\bar{\beta} \\ & \beta^2 & \sqrt{2}\bar{\alpha}\beta & \bar{\alpha}^2 \end{pmatrix} = \begin{pmatrix} 1 & \\ & \phi_1(A) \end{pmatrix} \tag{118}$$

where we have used $|\alpha|^2 + |\beta|^2 = 1$. This means a well–known decomposition

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1.$$

Let us take an analogy. For

$$V = \begin{pmatrix} X_0 & -Y_0^\dagger \\ Y_0 & X_{-1} \end{pmatrix}$$

we have

$$V \otimes V = \begin{pmatrix} X_0^2 & -X_0 Y_0^\dagger & -Y_0^\dagger X_0 & Y_0^\dagger Y_0^\dagger \\ X_0 Y_0 & X_0 X_{-1} & -Y_0^\dagger Y_0 & -Y_0^\dagger X_{-1} \\ Y_0 X_0 & -Y_0 Y_0^\dagger & X_{-1} X_0 & -X_{-1} Y_0^\dagger \\ Y_0 Y_0 & Y_0 X_{-1} & X_{-1} Y_0 & X_{-1}^2 \end{pmatrix}.$$

However, the analogy breaks down at this stage because of the non–commutativity

$$T^\dagger (V \otimes V) T \neq \begin{pmatrix} \mathbf{1} & \\ & \Phi_1(V) \end{pmatrix} \tag{119}$$

for (90). We leave it to the readers. There is no (well–known) direct metnod to obtain $\Phi_1(V)$ at the current time.

Last, let us make a commemnt. For the matrix T coming from the Clebsch–Gordan decomposition (see [11])

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

it is not difficult to see

$$\begin{aligned} & T^\dagger(A \otimes A \otimes A)T \\ = & \begin{pmatrix} \alpha & -\bar{\beta} & & & & & & \\ \beta & \bar{\alpha} & & & & & & \\ & & \alpha & -\bar{\beta} & & & & \\ & & \beta & \bar{\alpha} & & & & \\ & & & & \alpha^3 & -\sqrt{3}\alpha^2\bar{\beta} & \sqrt{3}\alpha\bar{\beta}^2 & -\bar{\beta}^3 \\ & & & & \sqrt{3}\alpha^2\beta & (|\alpha|^2-2|\beta|^2)\alpha & -(2|\alpha|^2-|\beta|^2)\bar{\beta} & \sqrt{3}\bar{\alpha}\bar{\beta}^2 \\ & & & & \sqrt{3}\alpha\beta^2 & (2|\alpha|^2-|\beta|^2)\beta & (|\alpha|^2-2|\beta|^2)\bar{\alpha} & -\sqrt{3}\bar{\alpha}^2\bar{\beta} \\ & & & & \beta^3 & \sqrt{3}\bar{\alpha}\beta^2 & \sqrt{3}\bar{\alpha}^2\beta & \bar{\alpha}^3 \end{pmatrix} \\ = & \begin{pmatrix} \phi_{1/2}(A) & & \\ & \phi_{1/2}(A) & \\ & & \phi_{3/2}(A) \end{pmatrix}. \end{aligned} \tag{120}$$

This means a well–known decomposition

$$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = (0 \oplus 1) \otimes \frac{1}{2} = \left(0 \otimes \frac{1}{2}\right) \oplus \left(1 \otimes \frac{1}{2}\right) = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}.$$

E Calculation of Some Integrals

We show some integrals.

(A) Compact case :

$$< f|g >= \frac{2(2j+1)}{2\pi}\int_{\mathbf{C}} \frac{d^2z}{(1+|z|^2)^{2j+2}} f(z)\overline{g(z)} = \sum_{k=0}^{2j} \frac{1}{{}_{2j}C_k} a_k \bar{b}_k \tag{121}$$

for $f(z) = \sum_{k=0}^{2j} a_k z^k$ and $g(z) = \sum_{k=0}^{2j} b_k z^k$ in $\mathcal{H}_J$.

This is reduced to the equation

$$\frac{2(2j+1)}{2\pi}\int_{\mathbf{C}} \frac{d^2z}{(1+|z|^2)^{2j+2}} z^k \bar{z}^l = \delta_{kl}\frac{1}{{}_{2j}C_k}.$$

If we use the change of variables

$$x = \sqrt{r}\mathrm{cos}\theta, \quad y = \sqrt{r}\mathrm{sin}\theta \quad \Longrightarrow \quad d^2z = dxdy = \frac{1}{2}drd\theta$$

then by using integration by parts

$$\begin{aligned}
\text{Left hand side} &= \delta_{kl}(2j+1)\int_0^\infty \frac{r^k}{(1+r)^{2j+2}}dr \\
&= \delta_{kl}(2j+1)\frac{k}{2j+1}\int_0^\infty \frac{r^{k-1}}{(1+r)^{2j+1}}dr \\
&= \cdots \\
&= \delta_{kl}(2j+1)\frac{k}{2j+1}\frac{k-1}{2j}\cdots\frac{1}{2j-k+2}\frac{1}{2j-k+1} \\
&= \delta_{kl}\frac{k!}{(2j)(2j-1)\cdots(2j-k+1)} \\
&= \delta_{kl}\frac{1}{{}_{2j}C_k}.
\end{aligned}$$

(B) Non–compact case :

$$< f|g >= \frac{2(2j-1)}{2\pi}\int_D d^2z(1-|z|^2)^{2j-2} f(z)\overline{g(z)} = \sum_{n=0}^{\infty} \frac{n!}{(2j)_n} a_n \bar{b}_n \tag{122}$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in H^2.

This is reduced to the equation

$$\frac{2(2j-1)}{2\pi}\int_D d^2z(1-|z|^2)^{2j-2} z^k \bar{z}^l = \delta_{kl}\frac{k!}{(2j)_k}.$$

Similarly in the case of (A), we obtain

$$
\begin{aligned}
\text{Left hand side} &= \delta_{kl}(2j-1)\int_0^1 (1-r)^{2j-2} r^k dr \\
&= \delta_{kl}(2j-1)\frac{k}{2j-1}\int_0^1 (1-r)^{2j-1} r^{k-1} dr \\
&= \cdots \\
&= \delta_{kl}(2j-1)\frac{k}{2j-1}\frac{k-1}{2j}\cdots\frac{1}{2j+k-2}\frac{1}{2j+k-1} \\
&= \delta_{kl}\frac{k!}{(2j)(2j+1)\cdots(2j+k-1)} \\
&= \delta_{kl}\frac{k!}{(2j)_k}
\end{aligned}
$$

by using integration by parts.

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